CONCEPTS OF FINANCIAL MATURITY

OF TIMBER AND OTHER ASSETS

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OPENING SUMMARY

This study investigates the problem of finding financial maturity for any appreciating asset, with especial but not exclusive attention to timber.

The problem may be likened to one of when to adjourn a convention. There are two elements of urgency prompting us to close the meetings: to release the men; and to release the convention hotel space they preempt. The problem is to balance these costs at the margin of decision, against the benefits of prolonging the meetings, and to arrive at an optimal hour of adjournment.

The solution is elusive because in practice the "hotel space" -- the site -- often has no predetermined cost, but must be imputed one in the course of solving the problem. This calls for a simultaneous solution, jointly determining site rent and financial maturity. This study works out the simultaneous solution using marginalist techniques and shows it to be identical with the classic Faustmann formula of forest economics.

The study then criticizes other concepts of financial maturity advanced by economists and foresters. Their fault is in failing to allow for both elements of urgency. The Chart on page viii lays out the various solutions considered, in their relationship to the two elements of urgency, site rent and interest rate. Only Faustmann's solution and the variant in the box just below it incorporate both elements of urgency in the solution.

Allen's and Fisher's "maximum discounted yield" allows nothing for the second element, value of release of the site, and drags the convention on too long. The foresters, Duerr, Guttenberg, and Fedkiw, raise the question of whether this omission makes enough difference in practice to warrant incorporating site rent into the solution. The study undertakes to demonstrate from analysis of standard forest yield data that it does.

The study develops and demonstrates an easily operable technique for incorporating site rent into the determination of financial maturity. Using this technique to analyze standard forest yield data, it concedes to Duerr, Guttenberg, and Fedkiw that the influence of site rent is sometimes negligible but finds that it is also sometimes considerable. It specifies and discusses the conditions under which site rent does affect financial maturity appreciably, concluding that these conditions obtain in many areas, and are likely to extend their sway in the future. It notes that site rent is much more important in nonforest determinations of financial maturity.

Boulding's maximum "internal rate of return" also fails to deal with the second element of urgency, but in the process imputes its value to the first, overstates the sum, and adjourns the meetings prematurely.

The paper discusses the choice among Faustmann's, Allen's and Fisher's, and Boulding's solutions. It finds the advantage of Faustmann's in its dealing adequately with both elements of urgency and discusses how the rejected solutions, especially Boulding's, may be partially salvaged within the framework of Faustmann's formulation. This produces a new concept of financial maturity, joint maximization of site rent and internal rate of return, that is recommended for limited circumstances.

Next the study criticizes zero-interest doctrines, which dismiss the first element of urgency, internal rate of return, and looks for reasons why such obviously indefensible doctrines are tolerated by many foresters.

The study then elaborates Faustmann's formula to deal with intermediate costs and revenues and suggests how to generalize the formula beyond the confines of forestry, beyond the limitations of appreciating assets in general, to find financial maturity of depreciating assets and contribute to the accurate solution of all economic problems of replacement and turnover.

Finally, the study applies its analysis to a number of practical questions of private and public policy. It concludes that forest rotations in the United States are on the whole uneconomically long, through inadequate recognition of one or the other element of urgency. It implicitly suggests improvement through wider adoption of Faustmann's formula and removal of institutional obstacles to its application. It notes that wide adoption of the formula, outside forestry as well as in, would probably tend to accelerate the turnover of the economy's capital stock, with significant macro-economic effects.

CHART: SOME CONCEPTS OF FINANCIAL MATURITY AND THEIR ADVOCATES IN RELATION TO TWO ELEMENTS OF URGENCY

If the Site Rent Is → / If the Interest Rate Is Equal to ↓	Zero	Annual Equivalent of Periodic Net Revenue
	Financial Maturity Is When	
Zero	Total Growth is Maximum (No Avowed Advocates)	Mean Annual Net Growth (Waldrente) is Maximum (Borggreve)
Market Rate	Discounted Net Revenue is Maximum (Allen and Fisher)	Annualized Net Revenue (Bodenrente) → is Maximum (Faustmann)
Maximum Internal Rate	"Internal Rate of Return" is Maximum (Boulding)	Site Rent and Internal Rate of Return Are Jointly Maximized (Present Study)

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CHAPTER I

INTRODUCTION: CONFLICTING CONCEPTS OF FINANCIAL MATURITY

Economists of several generations have relished a problem that begins, "Suppose I lay down wine in a cave to mellow ..." In large part the problem is "When should I take it out again?" Essentially the same analytical problem arises when one asks when to market livestock or harvest timber. Similar problems arise in deciding when to demolish old buildings, clear old orchards, scrap machinery, and clear out inventories.

This study presents what the writer considers a basically correct solution, together with criticism of other concepts of financial maturity advanced by economists R.G.D. Allen, Irving Fisher, Kenneth Boulding, Friedrich and Vera Lutz, and Clifford Hildreth, and foresters William A. Duerr, Sam Guttenberg, John Fedkiw, Bernard Borggreve, and Richard McArdle and Edward C. Crafts, the last two representing the official viewpoint of the Forest Service of the U. S. Department of Agriculture. The study devotes primary attention to practical applications of its solution in forestry, and incidental attention to generalizing it for application to all problems of turnover and replacement.

Before launching into any extended study of the question, one may fairly ask if the answer is not, as the mathematicians like to say, trivial? [Is it not obvious that timber should stand as long as, and no longer than, it is yielding the owner as good a percentage return as his assets could earn elsewhere?] Symbolically, where $g(t)$ is the growth function of the value of a stand of timber; Δg, the annual growth of value; and i, the relevant interest rate, does not financial maturity arrive when $\Delta g/g = i$? That is, we shall see, the answer of several economists. It bears the appearance of a forthright marginalist solution, where Δg is the incremental growth of a year's time and gi its incremental cost.

1/ The writer is obliged to the Ford Foundation for a grant of uncommitted research funds to facilitate this work. He owes thanks to Ralph Bryant, Ewald Maki, Rudolf Grah, and John Zivnuska for advice on forest technology, terminology, and literature; to Matthew P. Gaffney, Jr., for stimulating discussions of the mathematical concepts involved; to Thomas Martinsek for taking an interest in helping wrestle with some of the solutions; to George Morton and Lee Martin for valuable criticism; and above all to Dean C. Addison Hickman for his initiative in fostering an intellectual environment congenial to fundamental research. None of these is implicated in the result.

Its fault, however, is in omitting part of the incremental cost of time, of which gi, interest on the realized value of timber, is not the whole. Again, one may ask, is it so difficult simply to add these other incremental time-costs to gi? Basically that is indeed what one must do. But the operation is not so simple.

That is because one cost -- the annual value of the site the timber occupies -- is not as a rule an externally "given" datum, but is to be found in the very process of finding financial maturity. The best alternative use of timberland is not, unless it is submarginal for timber, some nonsylvan use. Rather the best alternative is to harvest the present stand and start the next. One cannot specify the value of this alternative without knowing the age of harvest, which affects it. So financial maturity depends on the annual value of the site, which in turn depends on financial maturity.

This sort of problem, of course, calls for a simultaneous solution. While this cannot be considered higher mathematics, still the process of formulating the simultaneous equations causes perplexities -- perplexities that have never been resolved, so far as the writer knows, in the literature of economics.

When one surveys the relevant literature of economics and forestry, one finds divided counsel indeed. Advocates there are for a number of solutions, many of them plausible enough until

Table 1. Optimum Rotations for European Larch on Site II Indicated by Various Criteria a/

Criterion Maximized	Rotation (years)
Boulding's internal rate of return	33
Faustmann's soil expectation value	48
Allen's discounted net yield	66
Borggreve's forest rent	80
Tree growth	Over 100

a/ Thinnings and intermediate costs disregarded. Interest figured at 2 per cent and regeneration costs at £10 per acre.

Source of Yield Data: W. E. Hiley, Economics of Forestry (Oxford: The Clarendon Press, 1930), 127.

Knut Wicksell wrote:

> If, in such a simple case, we are able to deduce the general laws of capital and interest, this deduction may be regarded as an essential ingredient in the explanation of all the more complex phenomena of actual employment of capital. 1/

On the other hand, so long as economic analysis fails to master this problem, it constitutes not only a failure, but, as no problem is an island unto itself, a nuisance and perhaps menace to the whole of economic theory. For most of the rival solutions mentioned above clash, not only with each other, but with general principles basic to much of economics: Boulding explicitly disavows the marginalist approach; Allen and others implicitly dismiss from their reckoning the annual value of land; some forest economists disavow the use of compound interest, or of any interest at all.

Such division is a challenge, too, to the practicing economist. On a valid concept of financial maturity rest key decisions in many important industries. This study emphasizes

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1/ Knut Wicksell, Lectures on Political Economy, Vol. I, General Theory, trans. E. Classen (New York: The Macmillan Company, 1934), 127.

forestry. As some 600-700 million acres 1/ of the land area of the United States are in timber, about one-third of its total, this constitutes a practical problem of some dimensions.

But the analytical problem is quite general. There comes a time to replace machinery, market livestock, demolish buildings, clear out inventories, or what you will, in almost every conceivable industry. Timber and many other biological assets differ from most others in that they appreciate. But we will see that a valid concept of financial maturity may easily be adapted to deal with depreciating assets as well.

We proceed as follows. In Chapter II we submit the elements of what the writer considers a correct solution, long known to foresters as the Faustmann formula, and put it in its most operable form. In Chapter III we criticize some incompatible concepts of financial maturity. In Chapter IV we elaborate the Faustmann formula and adapt it to cope with more complex and dynamic conditions, including nonforest problems. In Chapter V we draw from the Faustmann solution some of its more important practical implications. In Chapter VI we list the contributions of this study, and in Chapter VII some suggestions for future research, in forestry and in general.

1/ P. L. Buttrick, Forest Economics and Finance (New York: John Wiley and Sons, Inc., 1943), 147.

CHAPTER II

ELEMENTS OF THE FAUSTMANN FORMULA

Suppose, to keep time-preference within temperate bounds, we forsake the wine cellar and take up timber culture. We plant timber lands at the start of year one. Assume, for stark simplicity, that there are no intermediate outlays or revenues between planting and harvesting.

After ten years we note that the stumpage -- defined as the value of trees on the stump, net of harvest costs -- has grown to equal the original planting costs. In a few more years the stumpage equals the planting costs compounded (at the rate of interest we can earn on alternative investments). We begin to wonder when to harvest.

We recall that we should not wait longer than the time when annual growth (Δg) equals interest on the trees (gi); but that we probably should not wait even that long, because the annual value of the site, even though we do not lay out explicit payments for it each year, is part of the incremental cost of time. We recall that the annual value of the site in forestry itself depends on the year of harvest, posing a problem requiring simultaneous solution.

There are several perfectly good paths to the solution. Almost all, however, presuppose an understanding of the basic formula for annualizing a sum received after several years. A simple average per year will not do, even though Boulding has lapsed into this error. 1/ There is interest to consider as well as averaging. The formula must reckon with the contrast in the time-distribution of a lump sum and an annual payment spread over the years.

The annual equivalent of a sum received at the end of t years is that amount which, received annually, and accumulated along with the compound interest on it, will grow in t years to equal that sum. Symbolically 2/ where A is the sum received after t years, and a is its annual equivalent received at each year-end,

$$a = \frac{Ai}{(1+i)^t - 1} \qquad a = \frac{A}{\frac{(1+i)^t - 1}{i}} \qquad (1)$$

1/ Kenneth Boulding, Economic Analysis (3d ed.; New York: Harper & Bros., 1955), 871. Hildreth seems to have made a similar error although coming at it indirectly. See Clifford G. Hildreth, "Note on Maximization Criteria," Quarterly Journal of Economics, 61: 156-164, November, 1946, discussed on p. 58-59, below.

2/ For a list of algebraic symbols in this work, see Appendix to Chapter II, pp. 12-15

This is derived by summing the series,

$$A = a + a(1+i) + a(1+i)^2 + \ldots + a(1+i)^{t-1}$$

To find general algebraic solutions, it is convenient to deal with continuous functions. We will assume that *a* is received continuously (rather than at each year-end, as above). This assumption alters the above formula only in one particular: i in the numerator (not the denominator) is replaced by ρ, a figure almost equal of *i*, but minutely smaller.1/ The basic annualizing equation becomes,

$$a = \frac{A\rho}{(1+i)^t - 1} \quad \text{2/} \qquad (1a)$$

In the present problem, *A* is the net value product of the forest, received after several years. It is the excess of stumpage over compounded regeneration costs. Let g(t) represent the stumpage value; let C_o represent the regeneration cost, then

$$A = g - C_o(1+i)^t \qquad (2)$$

Net Revenue at time t.

and a, the annual value of the forest floor, is the annual equivalent of *A*:

annual rev. to NRt

$$a = [g - C_o(1+i)^t] \frac{\rho}{(1+i)^t - 1} \qquad (3)$$

Having thus expressed a as a function of t, we have the second equation we need to find the two unknowns. We simply substitute this definition of a in the equation of incremental product and incremental cost of time. This latter equation we have expressed in crude discontinuous form as

$$\Delta g = gi + a$$

g(t)i + a — annual payment for the interest on present value all for 1 unit period.

1/ For a proof of this substitution see Harry Waldo Kuhn and Charles Clements Morris, The Mathematics of Finance (Cambridge: Houghton Mifflin Co., 1926), 83-84.

ρ is often called the "force of interest" corresponding to a rate of interest, i. ρ is that rate which, when compounding is continuous, yields the same result as the use of i yields when compounding is annual. For most purposes ρ and i are interchangeable. For algebraic purposes ρ is the natural log of (1+i).

2/ We might also remove i from the denominator by replacing $(1+i)^t$ with $e^{t\rho}$. But this is not essential. The two expressions are equal by definition for all values of t and completely interchangeable. The use of $(1+i)^t$ does not necessarily imply that compounding is not continuous, but only that the rate specified, i, is the annual equivalent of whatever continuous rate, ρ, is used. To obviate a needless step and a less familiar form, we leave $(1+i)^t$ as is, with apologies to mathematical purists. Actually it serves a useful function to leave it unchanged. It emphasizes that continuous receipt of a and continuous compounding of interest are two distinct operations.

Putting it in continuous form, Δg becomes dg/dt, and i becomes ρ.

$$\frac{dg}{dt} = g\rho + a \quad 1/ \tag{4}$$

Substituting equation (3) for a, and abbreviating dg/dt as g',

$$g' = g\rho + [g - C_o(1+i)^t] \frac{\rho}{(1+i)^t - 1} \tag{5}$$

Solving for t we get a preliminary expression for the optimum rotation -- "preliminary" because g, on the right side, is still a function of t:

$$t = \frac{1}{\rho} \ln \frac{g'}{g' - \rho(g-C_o)} \tag{6}$$

While they do not write or derive it this way, or conceive of it as a marginalist solution, this is the foresters' Faustmann formula.

Doubtless many readers will find it more meaningful to visualize this solution as illustrated in Figure 1 (page 8). We want to maximize the annual equivalent, a, of the excess of growth over compounded regeneration cost, $C_o(1+i)^t$. This last is the curve marked δ, and the excess of growth is the difference between δ and the growth curve, g.

To maximize a we construct a curve, T, that exceeds δ by the cumulated sum of an annuity, a, compounded regularly at i, a given interest rate:

$$T = C_o(1+i)^t + a \frac{(1+i)^t - 1}{\rho} \tag{7}$$

To find the year of maximum a we elevate this curve T until it is tangent to g. This is the highest value of a possible within the given growth curve.

Algebraically we would find this tangency by requiring simultaneously that the T and g curves have equal ordinates, and equal slopes. Those conditions give:

$$g = T = C_o(1+i)^t + a\left[\frac{(1+i)^t - 1}{\rho}\right] \tag{7a}$$

$$g' = T' = C_o\rho(1+i)^t + a(1+i)^t = (C_o\rho + a)(1+i)^t \tag{7a'}$$

1/ We also apologize to purists for writing g(t) simply as g. The hope is that no one will forget that g is a function of time, and that the lessened clutter will make the equations more readable.

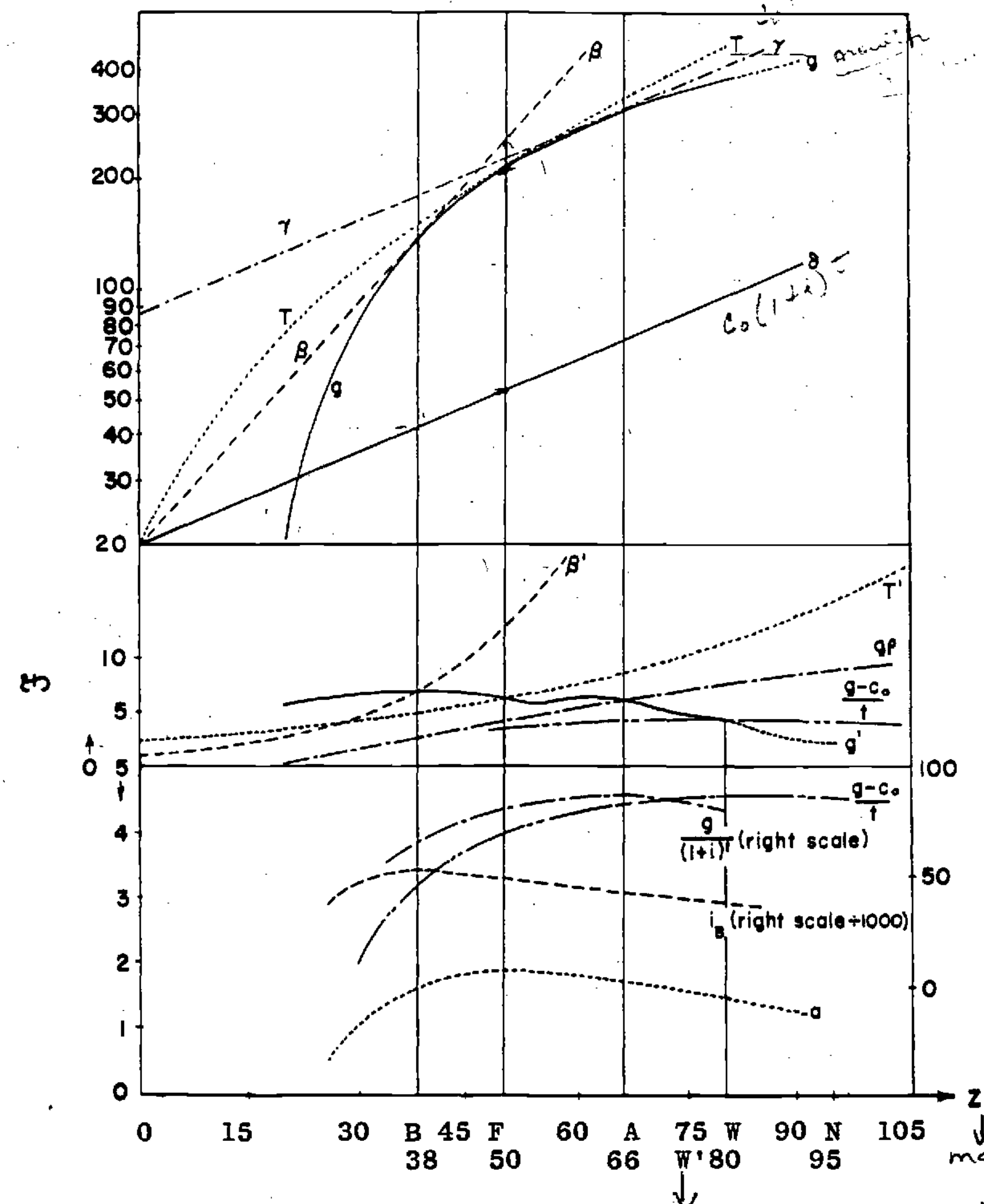

Figure 1. Growth Function (g) of Stumpage Value of European Larch on Site II, with Several Derived Functions Used in Finding Different "Optimum" Rotations. For a complete explanation of the relations and operations shown here, see the Appendix to Chapter II, pp. 12-15. The original data are from Hiley, W.E., The Economics of Forestry (Oxford: The Clarendon Press, 1930), p. 127.

Solving (7a) for a, and substituting in (7a'), we get

$$g' = \frac{\rho(1+i)^t (g-C_o)}{(1+i)^t - 1} \qquad (5a)$$

But this is another way of writing equation (5). Solving for t we get as before:

$$t = \frac{1}{\rho} \ln \frac{g'}{g' - \rho(g-C_o)} \qquad (6)$$

Faustmann's formula may also be derived in several other ways. As it is of prime importance to this study, several other proofs are shown in Appendix A, pp. 90-93. Anyone using the formula much would do well to master them all, as each adds something to one's understanding of the formula and hence enhances his ability to adapt it effectively to various circumstances. Without this flexibility the formula would probably be more liability than asset.

It remains to put the formula in its most operable form. Note that our Faustmann formula, equation (6), is not as it stands a final solution for t, because g and g' on the right side are themselves functions of t. So long as we have no specific function for g(t), we cannot carry the solution further. Nor is it at all likely, even if we had such a function, that we could arrive at a precise solution for t; the logarithmic form of the equation makes that nearly hopeless. This is more true of growth functions fitted to empirical yield tables, functions that must almost certainly be quite complex, involving powers of t.

Duerr, Guttenberg, and Fedkiw have put the formula in a very usable form (although in their urge for still greater simplicity, as we will see, they have excised a vital part of it). They have solved equation (5a) for the constant ρ, rather than t, putting it in the form,

$$\rho = \frac{g'}{g - C_o} \cdot \frac{(1+i)^t - 1}{(1+i)^t} \quad 1/ \qquad (6a)$$

The second fraction on the right side,

$$\frac{(1+i)^t - 1}{(1+i)^t},$$

may readily be derived from a standard tabulated form, its complement $\frac{1}{(1+i)^t}$. This Duerr, Guttenberg and Fedkiw call the "correction factor". We will call it the "correction coefficient" and designate it as:

1/ William A. Duerr, John Fedkiw, and Sam Guttenberg, Financial Maturity: A Guide to Profitable Timber Growing, USDA Tech. Bul. No. 1146 (Aug ?

$$\Phi(t) = \frac{(1+i)^t - 1}{(1+i)^t} = 1 - \frac{1}{(1+i)^t} \qquad (13)$$

Note that Φ cannot be greater than one.

The first fraction on the right side of (6a), $\frac{g'}{g-C_o}$, is simply the time-rate of growth as a percentage of the stumpage, g, net of regeneration costs, C_o.

This form of the Faustman solution is especially easy to hold in mind and work with. It tells us that timber reaches financial maturity when $g'/(g-C_o)$, the rate of growth referred to a base that is net of harvest and regeneration costs, has fallen to equal ρ, the interest rate, divided by a tabulated correction coefficient, Φ. Tables of ρ/Φ are given in Appendix C. 1/

The forester, working within the constraint of, say, a 5 per cent interest rate, can reckon at a glance that financial maturity has arrived when

$$\frac{g'}{g - C_o} = \frac{.04879}{\Phi(t)}$$

The entire right side appears in Appendix C as a unitary figure. Using this analytical tool one can compute financial maturity with a minimum of operational problems. The process is illustrated in Table 2, and there explained in more detail. 2/

The knowing forester need hardly be reminded that this formula does not assure a correct forecast of g', nor does it include all the relevant factors. More and more variables are introduced as we proceed. 3/

1/ Tables of i/Φ are found in some standard works on the mathematics of finance, e.g., Kuhn and Morris (1926), Appendix Table VII, "The annuity whose present value is one." Our Appendix C gives ρ/Φ, whose value is slightly lower.

2/ See Chapter III, Section 1, d, i.

3/ Professor George Morton has pointed out to me another operable solution, one which practicing foresters may prefer in some circumstances. He points out that the year of financial maturity is unaffected by whether or not we compound regeneration cost. This is not immediately obvious, but seems to hold up mathematically.

This lets us drop the compounding term, $(1+i)^t$, following C_o in Equation (3), which defines the site rent which we wish to maximize. (We can also drop ρ, which is not a function of time and does not therefore affect the solution.)

3/ (Continued) This leaves us with a simpler expression to maximize:

$$M = \frac{g - C_o}{(1+i)^t - 1}$$

Tabulating the denominator as a reciprocal we can arrive at the maximum value of M, in any particular case, quite expeditiously, simply by trial-and-error approximation.

This formulation would have the disadvantage of being limited in its application to instances wherein all revenues and costs were concentrated at the end-points of rotations. Where there are intermediate costs and revenues, compounding factors would have to be reintroduced, which would destroy the simplicity, hence the advantage of this technique. Neither is the solution readily adapted to handle changes of data that occur in mid-rotation. It presents a certain danger, too, in that the expression being maximized has no significance in itself, but only coincidentally reaches its maximum simultaneously with the a of Equation (3). The danger is that values of M might erroneously slip into use as values of a, either by sheer carelessness or because an operator using the formula might think it had some normative value.

APPENDIX TO CHAPTER II

SUMMARY EXPLANATION OF FIGURE I:

SYMBOLS, CURVES AND THEIR CONSTRUCTION 1/

TOP GRAPH: The total product of time and the total costs of time according to different concepts of financial maturity. The tangencies determine financial maturity. Note that the ordinate is logarithmic.

g: The original growth function of the stumpage value of European Larch on English Site II. The other curves are derived from these basic data, plus the assumed constants specified below.

The other top-graph curves represent total costs, according to different concepts of financial maturity. These costs all include imputed or residual costs, and they differ in their treatment of these latter costs. Financial maturity is determined by the tangencies of the respective cost curves with the growth curve, g.

On this semi-log scale, the slope of the g-curve is the percentage rate of growth, g'/g.

Significant variations in the time-rate of growth, g', may be undetectable on the g-curve in its upper reaches. See discussion of g' below.

δ: This curve shows regeneration cost, C_o, compounded at a given rate of interest, i, in this case 2 per cent. $\delta = C_o(1+i)^t$. The Y-intercept is C_o, in this case £20.

This curve shows no imputed costs and yields no solution.

β: Regeneration cost, C_o, compounded at the maximum value of Boulding's "internal rate of return," i_B, (q.v.). In this case i_B equals 5.22 per cent, its maximum value, which is realized at age 38. $\beta = C_o(1+i_B)^t$. This is the total cost of time in Boulding's concept of financial maturity. The tangency of B with the g-curve gives Boulding's solution, B, in this case 38 years.

T: Regeneration cost, C_o, compounded at the market rate of interest, i, plus the maximum soil rent value, a_F, accumulated and compounded at i. In this case a_F = £1.925 (realized at age 50). $T = C_o(1+i)^t + a_F\frac{(1+i)^t - 1}{p}$. This is the total cost of time in Faustmann's concept of financial maturity. The tangency with the g-curve determines financial maturity, in this case 50 years. T is also the value of immature standing timber, net of the site, at any time. T shows the market value of immature trees, not for immediate harvest, but for holding to maturity. The immediate harvest value is g.

γ: The highest curve whose percentage rate of growth is the market rate of interest, i, which also touches the g-curve. This determines its own Y-intercept, which is the maximum discounted value on the g-curve, in this case £86, realized at age 66.

$$\gamma = £86(1+i)^t.$$

1/ For a fuller discussion [illegible]

γ is the total cost curve according to the Allen-Fisher concept of financial maturity. The tangency with the g-curve determines this solution, A. In this case it is 66 years.

MIDDLE GRAPH: The incremental product of time, and the incremental cost of time according to different concepts of financial maturity. The intersections determine financial maturity. They correspond to the tangencies in the top graph, as noted by the solid vertical lines. Note that the ordinate is NOT logarithmic.

g': dg/dt, the time-rate of change of g. There is no formula for this curve, which is derived directly from the data of the g-curve.
Note the double maximum of the g'-curve. It represents the effect of quality increment, coming on strongly after volume increment has started to decline. It was unintentionally exaggerated in drafting, but is genuine nonetheless. It is undetectable in the g-curve of the top graph because of the latter's vast range. This exemplifies the advantages of marginal analysis, which lets us magnify and isolate the factors necessary to a decision.

β': $d\beta/dt$, the time-rate of change of β. $\beta' = C_o \rho_{Bmax}(1+i_{Bmax})^t$. i_{Bmax} is the maximum value of Boulding's "internal rate of return," in this case 5.22 per cent. ρ_{Bmax} is the corresponding force of interest, in this case 5.10 per cent.
In Boulding's concept of financial maturity, β' is the marginal cost of time. Its intersection with g' determines financial maturity.

T': dT/dt, the time rate of change of T. $T' = (C_o\rho + a_n)(1+i)^t$. a_n is the maximum value of a, soil rent (q.v.). In this case a_n = $1.925, the maximum value reached at age 50.
In Faustmann's concept of financial maturity, T' is the marginal cost of time. Its intersection with g' determines financial maturity.
In the year of maturity, in this case 50, T' equals $g\rho + a$, the simplified expression of the marginal cost of time used more often in the text. At other points, T' is greater than $g\rho + a$. Since they are equal at the time of solution, however, either expression gives the same rotation age.

$g\rho$: Interest on the stumpage, g, at the instantaneous rate of interest, ρ. In Allen's and Fisher's concept of financial maturity, this is the marginal cost of time. Its intersection with g' determines financial maturity.
Alternatively, a proponent of Allen's and Fisher's solution might insist that γ' be designated the marginal cost of time. $\gamma' = \$.86\ (1+i)^t$. γ' gives the same solution as $g\rho$, but equals it only at the time of financial maturity. Elsewhere $g\rho$ is smaller.

$\frac{g-C_o}{t}$: The mean annual net yield, with interest rate = 0. This is Waldrente, or "forest rent."
In the Waldrente concept of financial maturity, this is the incremental cost of time and also the maximand. This is because when interest is assumed at zero, there is no incremental cost except the imputed cost, which is the very thing being maximized.
Thus this curve appears twice, once in the middle graph as the incremental cost of time, and again in the bottom graph as the maximand.
The corresponding total cost curve and tangency have been

omitted from the top graph, to avoid congestion of lines. This total cost would be $\frac{[g_W - C_o]}{t_W} t + C_o$. On rectangular coordinates this would make a straight line. On the logarithmic coordinates of the top graph it would start from its Y-intercept of C_o, rise more steeply than T, then flatten out and touch g to the right of T's tangency.

BOTTOM GRAPH: The expressions maximized by the various concepts of financial maturity. The respective maxima determine financial maturity.

The maxima correspond to the tangencies of the top graph, and the intersections of the middle graph, as indicated by the solid vertical lines.

Note how flat the curves are near their maxima. This does not mean that a correct finding of financial maturity is unimportant, but rather that this method averages any error over the entire rotation period, thus obscuring it. This illustrates the superiority of the incremental or marginal approach of the middle graph, where the economic penalties of error stand out more clearly.

$\frac{g-C_o}{t}$: The mean annual net yield, with interest rate, i, equal to zero. See discussion just above. Maximand of Waldrente solution.

Maximum value of £ 4.60 at age 80, designated W.

$\frac{g}{(1+i)^t}$: The value of g discounted at the market rate of interest, i. Maximized by Allen's and Fisher's solution. Maximum value of £ 86, at age 66, designated A.

The age of maturity is not changed by subtracting compounded regeneration costs, $C_o(1+i)^t$, from g in the numerator.

i_B: Boulding's "internal rate of return". $i_B = \left(\frac{g}{C_o}\right)^{1/t} - 1$.

Maximized by Boulding's concept of financial maturity. Maximum value of 5.22 per cent, at age 38, designated B.

It is not a true rate of return, as it allows no return to the forest site whatsoever before computing the rate of return on regeneration costs, C_o.

a: Annual soil rent. The annual equivalent of the yield, g, net of compounded regeneration cost. $a = \frac{i\{g - C_o(1+i)^t\}}{(1+i)^t - 1}$

Maximized by Faustmann's concept of financial maturity. Maximum value is £ 1.925, at age 50, designated F.

This is the solution advanced in the text as the correct one.

THE ABSCISSAS: Rotation ages as determined by different concepts of financial maturity.

B, age 38: Boulding's solution. $i_B = 5.22$ per cent; $\rho_B = \ln(1+i_B) = 5.10$ per cent.

F, age 50: Faustmann solution. a_n = £ 1.925.

A, age 66: Allen's and Fisher's solution. $\frac{g_{66}}{(1+i)^{66}}$ = £ 86.

W', age 75: Maximum mean annual growth. (Forest Service of U.S. Dept. of Agriculture).

W, age 80: Waldrente, maximum mean annual net growth. $\frac{g-C_o}{80}$ = £ 4.60.

N, age 95: Maximum net yield per rotation -- a point of reference only.

Z, age above 105: Maximum total growth.

ASSUMED VALUES OF CONSTANTS:

C_o = £ 20

i = 2 per cent

ρ = 1.98 per cent

Appendix Table 1. Data a/ (in £) on Which Curves in Figure 1 Are Based

t	g	T	ġ'	$\frac{g}{(1+i)^t}$	$\frac{g - C_o}{t}$	i_B	a	β̇'	T'	gρ	g'
20	20.6	77.1	29.8	13.8	.03	0.15		2.81	3.44	0.4	
25	50.7		32.8			3.8	0.55				6.2
30	83.1	114.0	36.2	45.8	2.01		1.14	4.67			6.6
35	117		40.0			5.2-		6.0		2.3	6.9
40	152	162.7	44.7	69.0	3.30	5.2	1.75	7.7	5.13		6.9
45	186		48.8			5.1		9.9	5.66		6.6
50	218	217.8	53.8	81.2	3.96		1.925	13.0	6.24	4.3	6.2
55	248		59.4					15.6	6.90		6.0
60	278	287.6	65.6	84.8	4.30	4.5	1.84	21.2		5.5	6.2
65	310		72.4	85.6							6.2
70	339	372.0	80.0	84.9	4.56		1.71		9.29	6.7	5.5
75	364		88.3		4.58						4.9
80	388	474.5	97.5	79.6	4.60	3.8	1.48	58.6	11.3	7.7	(4.6)
85	(410)		107.6		(4.58)						(3.5)
90	(425)		118.8		(4.50)						(2.8)
95	(438)		131.2								(2.5)
100	(449)		144.9								

a/ Data in parentheses are projected from the original data.

CHAPTER III

OTHER CONCEPTS OF FINANCIAL MATURITY

Next let us consider some of the rival concepts of financial maturity that have been advanced by economists and foresters. These are, along with the Faustmann solution, depicted on Figure 1 and summarized in Table 8 (pp. 67-68), where they are given symbols. (F) is the Faustmann formula; (A) is R.G.D. Allen's solution, that Allen cites from Irving Fisher, the maximum discounted net yield; (B) is Boulding's solution, the maximum "internal rate of return", (W) is the Waldrente or "forest rent" solution, quite popular in the theory and practice of forestry, the same as the Faustmann formula but using a zero interest rate; (W') is a variation of the Waldrente solution used by the Forest Service, U.S. Department of Agriculture, in which regeneration costs are dismissed; (N) is maximum net yield, presented only as a point of reference; and (Z) is maximum growth.

1. The Maximum Discounted Value (A)

a. R.G.D. Allen and Irving Fisher

The proposal is to maximize the discounted value of g, that is, $g/(1+i)^t$. It was propounded by R.G.D. Allen 1/ and Irving Fisher 2/ and accepted more conditionally by Knut Wicksell. 3/ Probably most economists would on first thought incline toward this, at least as a first approximation, which indeed it is.

It appears even more plausible when we set the derivative of the discounted value to equal to zero and obtain the maximizing condition,

$$g'/g = \rho \qquad (14)$$

This is to say one should hold his timber until its percentage rate of growth falls to the interest rate.

1/ R.G.D. Allen, Mathematical Analysis for Economists (London: Macmillan and Co., Ltd., 1933), pp. 248-250. Lutz attributes to Allen a totally different criterion, maximization of the output-input ratio (Lutz and Lutz, op.cit., p. 16 n). The citations the Lutzes give in Allen, pp. 362 ff. and pp. 404 ff., concern somewhat different problems. On pp. 248-250 Allen specified the wine and timber rotation problem, and maximizes $\frac{g}{(1+i)^t}$ to obtain the solution.

2/ Irving Fisher, The Theory of Interest (New York: The Macmillan Co., 1930), 161-165.

3/ Knut Wicksell, Lectures on Political Economy, Vol. I, General Theory, trans. E. Classen (New York: The Macmillan Co., 1934), 172 ff. Wicksell's treatment is ambiguous enough, at least in translation, so that one cannot number him with certainty among advocates of this solution.

But some implications of Allen's and Fisher's solution raise immediate doubts of its general validity. Suppose the interest rate, ρ, approaches zero, and regeneration costs, C_o, are zero: it seems self-evident that one should then aim to maximize annual growth, g/t. Time-distribution would be immaterial and one would simply maximize his annual income through time. This is solution W', maximum annual growth, to which Faustmann's solution reduces when both C_o and i equal zero. But Allen's and Fisher's solution moves far out beyond W'. It continues out to Z, the time of maximum growth, where g' equals zero. It maximizes output per rotation, rather than per year.

Another anomaly of this solution is that regeneration costs, C_o, however high or low, do not affect the rotation. In fact neither Allen nor Fisher states how regeneration costs would be handled. Presumably this is because they realized that subtracting compounded costs from stumpage would not affect the solution; one would compound C_o only to discount it, leaving it standing alone.

$$\frac{g - C_o(1+i)^t}{(1+i)^t} = \frac{g}{(1+i)^t} - C_o \qquad (15)$$

Not being a function of time, C_o would not move the maximum at all. But this seems wrong, as I will try to demonstrate. The annual burden of this cost may be reduced by longer rotations, and a higher C_o calls for longer rotations, ceteris paribus.

A third anomaly in Allen's and Fisher's solution is the unimputed excess, in year one, of discounted stumpage value over regeneration cost. There will be such an excess on all sites where timber yields anything above the recovery of regeneration cost with interest -- i.e., on all but marginal sites. Allen's and Fisher's solution implies that one need only plant seedlings on good sites to have their value as investments rise immediately to the maximum discounted stumpage. On Figure 1 this would mean a jump up the ordinate from £20, the regeneration cost, to £86, the y-intercept of the discounting curve Y. Were such immediate gains truly possible, every day would be Arbor Day with everyone multiplying his assets as fast as he could plant and sell.

The anomalies spring from treating timber as though it were a sort of redeemable bond. The general fault of Allen's and Fisher's solution is its not recognizing that timber stands on a site whose time-vector is part of the variable cost of adding growth. Faustmann's solution recognizes two elements of urgency spurring the forester to harvest: urgency to release capital for future uses, and urgency to release the site. Allen and Fisher account only for the first of these. We first derived Faustmann's formula by identifying the incremental cost of time as interest on stumpage, $g\rho$, plus the annual value of the site, a, and equating their sum with the incremental product of time, g'.

Allen's and Fisher's formulation, if thus derived as a marginalist solution, is lacking the a. Faustmann's solution derives from the maximizing condition,

$$g' = g\rho + a \tag{4}$$

while Allen's and Fisher's solution derives from

$$g' = g\rho \tag{14a}$$

The fault shows up clearly at the margin of decision, time F (year 50) on Figure 1. Here Faustmann's solution bids the forester harvest, for beyond here the annual cost of time exceeds the annual growth. But Allen's and Fisher's solution bids the forester postpone harvest, for it excludes the annual value of the forest site a from the annual cost of time. The dot-dash line represents the Allen-Fisher incremental cost of time, g . In Figure 1, which is based on real growth data, $g\rho$ comprises only 69 per cent of $(g\rho + a)$ at year 50. Following $g\rho$ out to its intersection with g' at year 66, we see it prescribes an extended excursion into later years when the annual cost of time substantially exceeds its annual product. 1/

So easy is it to fall into Allen's and Fisher's solution along several paths, it is well to scout out some of these. We will post three of the more appealing approaches.

First, Allen's and Fisher's solution appears to follow directly from the familiar proposition that one finds the present value of a future sum by discounting it: dividing it, that is, by $(1+i)^t$. An investor holding immature timber for future harvest would maximize the present worth of his investment by planning to select t so as to maximize $g/(1+i)^t$. This leads to Allen's and Fisher's solution. On Figure 1 one simply pushes line δ upwards until it becomes line γ, the highest curve whose rate of growth is ρ which touches g.

But that application of a familiar principle is not valid, because the future trees are the joint product of the present trees plus the forest site. What is worth $g/(1+i)^t$ is the present stand plus the use of the land under it from now to t. The stand alone is worth less. To be exact, where T_o is the value of the trees alone at year zero:

$$T_o = \frac{g - a\,\dfrac{(1+i)^t - 1}{\rho}}{(1+i)^t} \tag{7a}$$

1/ By year 66, the excess of annual cost over annual growth will in fact exceed a, as shown on pp. 90-93.

The second term in the numerator represents the value of the site from years zero to t. One may dismiss it only on marginal sites, where a equals zero. 1/

T_o, as defined in (7a), obviously reaches its maximum earlier than T_o as Allen and Fisher would define it. For the second term in the numerator grows with t. Setting the time-derivative of T_o equal to zero to find the maximizing condition we obtain:

$$g' = g\rho + a \qquad (4)$$

This is simply equation 4 again, the postulate from which we originally derived Faustmann's formula. 2/

A second approach leading toward Allen's and Fisher's solution is the idea that, since land's income derives from selling trees, the rotation of highest present tree value must also give the highest land value. The error here is in forgetting that land value derives not just from the first harvest, but from that plus all subsequent harvests. There is a benefit in bringing all these latter forward, a benefit we have expressed as the annual value, a, of releasing the site for future uses. One may also account for this by maximizing land value expressed as the sum of the present net values of all future harvests. It was thus that Faustmann originally arrived at his solution. (See Appendix A, pp. 90-93.

A third route to Allen's and Fisher's solution runs along this line: "Faustmann arbitrarily takes C_o as externally fixed, and maximizes a. It is equally valid to take a as arbitrarily fixed and maximize the present value of the stand." The major fault in this is that Allen and others have not assumed a to be externally fixed. They have overlooked it altogether, in effect assuming it at zero, which is quite another matter. The choice of Allen's and Fisher's versus Faustmann's solution is not just a matter of taste or circumstance. Faustmann does allow for C_o, with

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1/ Once the optimal harvest age is found, Equation (7a) shows the growth of the investment value of immature timber. Let m be the year of maturity and t any year. To find T_t, the value of timber in year t, subtract t from m in the exponents of (7a), holding g and a fixed at g_m and a_m. As t grows from zero to m, the equation shows the growth of the investment value of immature timber along a smooth curve from C_o up to g_m. This is the finely dashed curve marked T on Figure 1. On land leased or taxed at its full annual rent, immature timber would in a perfect market change hands at prices along the curve T. This curve is also found from equation 7, which compounds C_o and accumulates and compounds a from year zero forward.

2/ For more detail see Chapter III, Section 3, a below.

compound interest, and simply maximizes the residual return to land. But Allen and Fisher allow for no return to land whatsoever.

It is true that if one were to take a as externally fixed, he could then maximize the T_o of equation (7a) and reach a defensible solution, which would differ slightly from Faustmann's solution if the external a differed from the maximum residual a. One might also fix both C_o and a and maximize i. The choice among these would depend on individual circumstances, a being the most likely choice because the site is generally more narrowly specialized for forestry, especially in the long run, than the capital input, which is converted into money with each harvest. But none of these is Allen's and Fisher's solution. 1/

An externally fixed a would be appropriate on wooded land whose best use was non-sylvan. Here the external a would exceed the maximum residual a and lead to a harvest even earlier, and hence farther from Allen's and Fisher's, than Faustmann's formula would prescribe. The same reasoning applies to understocked stands. 2/

Allen's solution, then, is valid only on land of no value where stumpage yields are only enough to return planting costs with interest. Elsewhere it prescribes too long a rotation. For there is an urgency to release the site for future uses, of which Allen's analysis takes no account.

b. Friedrich and Vera Lutz

Friedrich and Vera Lutz, in Chapter II of their THEORY OF INVESTMENT OF THE FIRM, discuss several possible criteria of financial maturity under various assumptions. Two sets of assumptions eventuate in Fisher's and Allen's solution -- they identify it by Jevons' name. They also arrive at Faustmann's solution, but only as one of many possibilities. They leave the impression that "Jevons'" solution is perfectly valid under usual conditions, which they specify. Their coming to such a conclusion, despite their cognizance of Faustmann's solution, makes it important to consider their reasoning.

i. The Limited Planning Horizon (p. 27)

Here they assume the forester's planning horizon is only as

1/ See Chapter III, Section 3, a for fuller treatment of this point.

2/ See Chapter III, Section 1, d, ii and Chapter IV, Section 3.

long as the rotation period. They assert that this assumption lets one dismiss future plans and thus reach "Jevons'" solution by maximizing the present value of harvest revenue. That is, since the forester is interested in nothing beyond the first harvest, he selects a growth period such as to maximize the present value of that single harvest. This discounted harvest value is what Fisher and Allen also maximized, and of course yields the same solution.

If at harvest time we are not concerned about future plans, why do we charge any interest? It is vain to say we are maximizing the "present" value of the timber as of, say, 40 years ago -- at harvest time who cares about that? The only reason for charging interest is that there are anticipated future uses for the money tied up in the trees.

If we charge interest, we reveal that we are, after all, interested in future plans for our assets. And then we would also have to charge soil rent, or, if you prefer, interest on the capital value of the site, or again, as the Lutzes phrase it, "interest on the present value of future profits" (p.33).

Furthermore, if our horizon equals one rotation, then by harvest time the horizon extends all the way forward to the next harvest. And one rotation is all that we need to compute soil rent and thus know the alternative value of land for the next rotation.

ii. The Overlapping Infinite Chains -- i.e., Uneven Aged Stands (pp. 32-35)

After exploring the implications of several limiting assumptions, the Lutzes finally suppose that the forester's horizon is not arbitrarily limited. They then come to Faustmann's solution, using in fact one of the same derivations as Faustmann. They foresee an "infinite chain" of future rotations and maximize the present value of the infinite series of future harvests, net of compounded regeneration costs.

But they regard this as a special case, applying only to even-aged stands. They immediately revert to Jevons' solution by supposing the forester to prefer an uneven-aged stand, with many overlapping rotations growing together. They regard this assumption as "more appropriate" for the unlimited horizon assumption and conclude that "The interest on the present value of future profits 1/ drops out of the solution entirely." (p.33).

1/ That is, on land value, although they never use that term.

On its face this proposition is most implausible. If Faustmann's is the correct rotation for one "infinite chain" and the overlapping chains are independent in costs and revenues, as the Lutzes assume, and grown on other land, as of course each individual tree would have to be, what economic consideration changes the rotation? Do we get a different rotation by analyzing a group of unrelated problems jointly instead of severally? Is the choice of a rotation entirely arbitrary, depending on how the analyst happens to feel like treating it?

It seems likely that the Lutzes have made some error in aggregation. Certainly if we maximize the summed soil rents from two infinite chains, or five, or 22, or any specific number, we still get Faustmann's solution, since summing them involves only discounting each future one by a fixed discount factor, and adding. How then do the Lutzes arrive at their conclusion?

Their method is to make the number of overlapping rotations increase with the length of rotation. Thus, for example, if we start one rotation each year, the number of overlapping "infinite chains" equals the rotation age. Lengthening the rotation age from 30 to 35 means increasing the number of chains by five.

Proceeding from this assumption their algebraic manipulations are unexceptionable. They put a Faustmann soil expectation value on each "infinite chain" at its date of inception, discount each one back to that date, add them up and maximize. The expression they end up maximizing is simply the discounted net yield capitalized, and as the capitalizing factor, ρ, is not a function of time, this gives the same result as maximizing the discounted net yield alone, i.e., Fisher's and Allen's solution.

They have overlooked one vital detail, however. As the number of chains increases on a fixed land base, the space allotted to each must decrease. The yield from each chain would therefore also decrease as the rotation age increased, which in turn would tend to shorten rotation ages.

They assume, however, that yields grow with time just as before. This must mean they are expanding onto new land. They do not pay for it, however. If they did, they would of course have to find that a land input that increases as a function of the rotation age must tend to lower the optimum rotation age.

To be sure they never use the word "land" or "rent". What they maximize they call "profit", a term they never define and whose beneficiary they never identify. We have assumed that the mysterious residual imputee was the site. We should, however, entertain another possibility: that they intend to subsume the site with $\underline{C}_o$ as an input at $\underline{t}_o$. This is unlikely since they assume the $\underline{C}_o$ input is repeated in its entirety with each new rotation. The land input is in fact distinctive in that it need be applied only once. 1/

But suppose, even so, they intended to subsume the site with C_o. Then their "Jevons'" solution is much the same as Faustmann's since the base on which they are just earning a market return at harvest time includes the site value. Their method of reaching Faustmann's solution in this case is open to serious question, but there is little point in conjecturing at length on what they may or may not have meant by "profit" and other equivocal terms. It is clear that they have failed to establish Allen's and Fisher's solution. 2/

c. Nominal Cpportunity Costs for the Site

Another approach to Allen's solution, found frequently in practice, is achieved by dismissing the annual cost of land with a nominal figure. The writer has heard a forest management consultant for the Weyerhaeuser Timber Company and spokesman for the National Association of Manufacturers estimate the annual value of the company's timberlands at their alleged rental for sheep-grazing, a negligible figure. The Faustmann solution by contrast rests on the postulate that the best alternative use of timberland is growing the next crop of timber. As long as this opportunity cost is greater than any nonforest alternative, the use of the latter constitutes an understatement. Nonforest alternatives are in point only when more remunerative than forestry, in which case they would always lead to rotations shorter than the Faustmann, i.e., rapid clearing to release land for nonforest uses.

1/ See pp. 52-54 for further discussion of how land can be treated as an input.

2/ Hildreth, as long ago as 1946, in criticizing a previous paper in which Friedrich Lutz advanced this same proposition, seems to have had in mind the same shortcoming we have emphasized: ". . . the longer the period of production chosen, the larger will be the grove of trees in existence after full production is reached (Hildreth, 1946, p. 161) ". . . at no point does he (Lutz) take account of the fact that the rent charge would be higher for a long production period than for a short one." (Op. cit., p.156, n. 7). The Lutzes, in introducing their 1951 chapter, remark that they are expanding the earlier treatment in response to Hildreth's comments (p. 17). It is curious that they did not take cognizance of this most decisive criticism by Hildreth.

d. Sam Guttenberg, William A. Duerr, and John Fedkiw

i. Criticism of their Generalization and Demonstration of the Use of the Correction Coefficient ϕ.

An indirect approach to Fisher's solution is advanced by Sam Guttenberg of the Southern Forest Experiment Station. Guttenberg puts a premium on simplicity and speed of computation. His theme seems to be that Allen's and Fisher's solution -- which he calls "financial maturity" -- differs so little from Faustmann's that one might as well use the simpler concept. In one note 1/ Guttenberg went so far as to assert that the two methods yield "precisely the same answers", and for full measure he included Boulding's solution.

This is obviously untenable. But in a more guarded recent joint paper with Duerr and Fedkiw 2/ he has made his case less absolute and more persuasive. The three authors acknowledge Faustmann's solution as "mathematically and theoretically sound" 3/ and different from Allen's and Fisher's. They maintain, however, that in practice the difference between the two is almost always too slight to warrant the trouble of using Faustmann's, which they imply involves some additional computations.

Algebraically they express their point as follows. They write the Faustmann optimum conditions in the most operable form, for which we have already credited them:

$$\rho = \frac{g'}{g - C_o}\,\phi, \text{ where } \phi = \frac{(1+i)^t - 1}{(1+i)^t} \qquad (6a)$$

The correction coefficient, ϕ, they would dismiss from practical consideration, except in unusual circumstances where t and/or i are "very small" (which tends to augment the influence of ϕ). They are also inclined to drop regeneration costs, C_o, from the denominator. They correctly point out that this omission tends to offset the other: dropping ϕ lengthens the rotation, while dropping C_o shortens it. This leaves them with Allen's and Fisher's solution:

$$\rho = g'/g \qquad (14)$$

I believe, however, they have overstated the difficulty of applying the correction coefficient and understated the errors that may ensue from dismissing it. The next few pages purport to show, first, how one may use ϕ in practice about as easily as not

1/ Sam Guttenberg, "Financial Maturity Versus Soil Rent," Journal of Forestry, 51 (1953), 714.

2/ William A. Duerr, John Fedkiw, and Sam Guttenberg, Financial Maturity: A Guide to Profitable Timber Growing, USDA Tech. Bul. No. 1146 (August, 1956).

3/ Ibid., p. 31.

and second, that dismissing ϕ will ofttimes, although not always, introduce large errors and economic losses.

As to the first, one need only prepare a table of ρ/ϕ, as described above on pp. 9-11, and presented in Appendix C. Select the interest rate appropriate to the time and one's financial circumstances, just as one would for Allen's and Fisher's solution. Compute

$$\frac{g'}{g - C_0}$$

for a few years near the probable solution, just as one would for Allen's and Fisher's solution only at a somewhat lower age. Tabulate these values in a column. Then simply read down this percentage yield column until it falls to equal ρ/ϕ, which also falls with age, but not as fast. To find Allen's and Fisher's solution, one must also read down the percentage yield column until it equals ρ alone.

The only added operational difficulty in applying Faustmann's solution is that ρ/ϕ falls with time, whereas ρ alone remains constant. But this added burden seems negligible relative to the stakes and the ample time the patient forest allows us to ponder over its fate.

Furthermore, applying ϕ spares us the trouble of figuring up explicit annual costs, as is sometimes done. Applying ϕ automatically computes the maximum annual cost the site could bear and treats it as an implicit annual cost, site rent, in determining the rotation age. Where there are explicit annual costs, these simply reduce site rent by their own amount, hence leaving optimal rotations unchanged, 1/ assuming the costs are constant over time.

Applying ϕ similarly takes care of any constant annual revenues. Inconstant intermediate costs and revenues require further analysis, by any system. 2/

Examples of the use of ϕ are shown in Table 2. The first two columns are values of ρ/ϕ, at 2 per cent and 5 per cent, excerpted from Appendix C. The other columns are percentage yields of various species of commercial timber, computed by the writer from standard data in forestry literature. 3/ For lack of specific data I have

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1/ See Chapter IV, Section 1.

2/ Chapter IV, Sections 1 and 2, deal with this question.

3/ Yield tables usually give data only for each 5th or 10th year. This makes for minor ambiguities in the estimate of annual increment, g', in any year. After experimenting with several techniques, I adopted simple interpolation. This has the disadvantage of not allowing for curvature of the g' curve, which in some early ages would be significant. In a few instances, where a solution fell close to the beginning of a year, I awarded it to the earlier year to allow for curvature obscured by linear interpolation. But none of these, as it happened, found their way into the tables used in this study.

Table 2A. Percentage Growth Rates of Several Species of Commercial Timber, and of the Duerr Synthetic Function, with Values* of ρ/ϕ for i = 2 Per Cent and 5 Per Cent

Item	Unit	15	20	25	30	35	40	45	50	55	60	65	70	75	80
							(per cent)								
ρ/ϕ i = 2%		7.70	6.06	5.08	4.42	3.96	3.62	3.36	3.15	2.99	2.85	2.73	2.64	2.56	2.49
i = 5%		9.40	7.83	6.92	6.34	5.96	5.69	5.49	5.34	5.23	5.16	5.09	5.05	5.01	4.98
Duerr Synthetic Function a/									8.5	5.4	3.8	2.8	2.1		
Upland Oaks, Site I b/	cu.ft.	5.51	4.45	3.86	3.23	2.81	2.47	2.06	1.89						
Mass. Red Oaks Site c/	cu.ft.		9.74	6.66	4.72	3.64	3.61	3.42	2.58	2.01					
Vermont Hardwoods Site II d/	cords		4.42	3.64	3.07	2.46	2.06	2.07	1.49						
Yellow Poplar Site 120' e/	cu.ft.	14.2	8.4	6.0	4.5	3.57	2.96	2.58	2.16						
Slash Pine Site 90' f/	cords	7.5	3.90	2.70	2.22	2.0									
Ponderosa Pine Site 160' g/	cu.ft.		5.75	4.46	3.65	3.09	2.59	2.23	1.85						
Eastern Cottonwood Site h/	bd.ft.				11.5	2.6	0.6								
Jack Pine Site III Merchantable i/	cu.ft.		10.92	7.10	4.70	3.44	2.50	1.87							
Loblolly Pine Site 110' j/	cu.ft.		7.79	5.05	3.51	2.58	1.99								
Loblolly Pine Site 90' k/	cords	11.11	7.41	5.14	3.70	2.78	2.13	1.49							
Loblolly Pine Site 100' l/	bd.ft.			10.9	6.9	5.01	2.98	1.77							
Loblolly Pine Site 100'	dollars			15.5	14.0	9.8	5.37	3.43	2.25	1.45					
Redwood Site II U.S. m/	cu.ft.		6.27	5.29	4.37	3.49	2.66	2.09							
Redwood Site I England n/	cu.ft.		6.6	5.05	4.02	3.31	2.72	2.07	1.56	1.20					
Redwood Site II U.S.	bd.ft.			13.7	8.26	5.75	4.29	3.27	2.93	1.83					
European Larch Site II o/	cu.ft.		14.5	7.9	5.15	3.66	2.66	1.91	1.59	1.37	1.20				
European Larch Site II	£			12.2	8.0	5.9	4.54	3.55	2.84	2.42	2.23	2.00	1.62	1.34	
European Larch Site I	cu.ft.		10.4	5.8	3.90	2.86	2.14	1.62	1.33						
European Larch Site V	cu.ft.						(5.54)	4.33	3.56	2.79	2.12	1.83			
Douglas Fir Comprehensive Inv. p/	bd.ft.				6.81	4.65	3.30	2.48	2.05	1.73					
Douglas Fir Best Wood Only q/ Site 140'	bd.ft.											7.32	5.53	4.44	3.64

Table 2B. Optimal Rotation Ages in Years for Above Growth Functions, with and without Use of the Correction Coefficient, ϕ, at 2 Per Cent and 5 Per Cent

Item	Unit	Optimal Rotation Ages: At 2 Per Cent: Uncorrected (years)	Corrected (years)	Diff.	% increase	At 5 Per Cent: Uncorrected (years)	Corrected (years)	Diff.	% increase
Duerr Synthetic Function a/		71	66	5	8	56+	56-	0+	0+
Upland Oaks, Site I b/	cu.ft.	47	10	37	370	18	10	8	80
Mass. Red Oaks Site c/	cu.ft.	55	46	9	20	29	24	5	21
Vermont Hardwoods Site II d/	cords	45	20	25	125	20	20	?	?
Yellow Poplar Site 120' e/	cu.ft.	55	44	11	25	35	31	4	13
Slash Pine Site 90' f/	cords	35	15	20	133	18	14	4	29
Ponderosa Pine Site 160' g/	cu.ft.	48	20	28	140	23	20	3	15
Eastern Cottonwood Site h/	bd.ft.	37	34	3	9	73	33	0	0
Jack Pine Site III Merchantable i/	cu.ft.	44	32	12	37	29	25	4	16
Loblolly Pine Site 110' j/	cu.ft.	40	25	15	60	25	20	5	25
Loblolly Pine Site 90' k/	cords	41	26	15	58	25	20	5	25
Loblolly Pine Site 100' l/	bd.ft.	44	38	6	16	35	32	3	9
Loblolly Pine Site 100'	dollars	52	46	6	13	41	40	1	2.5%
Redwood Site II U.S. m/	cu.ft.	45	29	16	55	27	20	7	35
Redwood Site I England n/	cu.ft.	45	25	20	80	25	20	5	25
Redwood Site II U.S.	bd.ft.	54	45	9	20	38	34	4	12
European Larch Site II o/	cu.ft.	44	33	11	33	31	27	4	15
European Larch Site II	£	65	45	20	44	38	35	3	9
European Larch Site I	cu.ft.	41	27	14	52	27	24	3	13
European Larch Site V	cu.ft.	62	53	9	17	42	39	3	8
Douglas Fir Comprehensive Inv. p/	bd.ft.	51	38	13	34	34	31	3	10
Douglas Fir Best Wood Only q/ Site 140'	bd.ft.	102	95	7	7	73	72	0	0

Table 2C. Rotation Ages, Corrected and Uncorrected, at 2 Per Cent and 5 Per Cent, Showing Increase of Rotations Due to Lower Interest Rates

Species	At 5% (years)	At 2% (years)	Increase from 5% to 2% (years)	Per Cent Increase
Upland Oak b/				
Uncorrected	18	47	29	161
Corrected	10	10	-	-
Mass. Red Oak c/				
Uncorrected	29	55	26	90
Corrected	24	46	22	92
Vt. Hardwoods d/				
Uncorrected	20	45	25	125
Corrected	20	20	-	-
Yellow Poplar e/				
Uncorrected	35	55	20	57
Corrected	31	44	13	42
Slash Pine f/				
Uncorrected	18	35	17	94
Corrected	14	15	1	7
Ponderosa Pine g/				
Uncorrected	23	48	25	109
Corrected	20	20	-	-
Jack Pine i/				
Uncorrected	29	44	15	52
Corrected	25	32	7	28
Loblolly Pine S. 110' j/				
Uncorrected	25	40	15	60
Corrected	20	25	5	25
Redwood S. II U.S. m/				
Uncorrected	27	45	18	67
Corrected	20	29	9	45
Eur. Larch II,cu.ft. e/				
Uncorrected	31	44	13	42
Corrected	27	33	6	22
Douglas Fir, S. 140' q/				
Uncorrected	34	51	17	50
Corrected	31	38	7	23

Table 2 footnotes:

* For complete tables of ρ/ϕ see Appendix C.

Percentage yields in Table 2 are computed from primary yield data published in the following sources:

a/ Synthetic function. Duerr et al., 1956, p. 33.
b/ Upland Oaks. Forbes, p. 21, cited from G. L. Schnur, 1937.
c/ Massachussetts Red Oak. Forbes, p. 1, cited from R. T. Patton, 1922.
d/ Vermont hardwoods. Forbes, p. 19, cited from A. F. Hawes, et al., 1914.
e/ Yellow Poplar. Forbes, p. 43, cited from E. F. McCarthy.
f/ Slash Pine, Forbes, p. 39, cited from USDA Miscellaneous Publication No. 50, 1929.
g/ Ponderosa Pine. Forbes, pp. 26-32, cited from W. H. Meyer, 1938, USDA Technical Bulletin No. 630.
h/ Eastern Cottonwood. Forbes, p. 2, cited from A. W. Williamson, 1913.
i/ Jack Pine. Forbes, p. 23, cited from S. R. Gevorkiantz, 194[illegible]
j/ Loblolly Pine. MacKinney and Chaiken, 1939, Table 10.
k/ Loblolly Pine. Forbes, p. 24, cited from USDA Miscellaneous Publication No. 50, 1929.
l/ Loblolly Pine. Davis, p. 235. Davis worked out the monetary yield table from data in W. H. Meyer, 1942, "Yield of even-aged stands of Loblolly Pine in Northern Louisiana," Yale University School of Forestry Bulletin No. 51.
m/ Redwood. Forbes, p. 44, cited from Donald Bruce, California Agricultural Experiment Station Bulletin No. 361, 1923.
n/ Redwood. Hiley, 1930, p. 245.
o/ European Larch. Hiley, 1930, pp. 127, 241.
p/ Douglas-Fir. Forbes, p. 3, cited from F. X. Schumacher, 1930.
q/ Douglas-Fir. McArdle, 1930, p. 67.

had to assume C_o at zero, which gives a bias to be discussed presently.

One finds optimal rotation ages simply by matching the percentage yields with values of ρ/ϕ . Rotation ages so determined are given in the lower part of Table 2.

As to the second point, it is not at all certain that the effect of dropping ϕ is usually small, as Duerr, Guttenberg and Fedkiw intimate. The truth of the proposition depends on the actual shape of yield functions, derived from field observation. To test it one would have to compute Fisher's and Faustmann's solutions for large numbers of such functions. But this the joint authors have not done.

Instead, they present a synthetic growth function,

$$g = -0.15t^2 + 29.5t - 930 \text{ (in dollars)},$$

for which the rotations differ by only two to three years in rotations of about 80 years. On this one numerical example they base their entire case. 1/

Therein lies its failing. One can find some growth functions that resemble theirs, but one also finds many that do not; therefore, their solution lacks generality. Their function is distinctive in that its percentage rate of growth, g'/g, passes through the range of customary interest rates, say 3 per cent to 2 per cent, at a more advanced age than do many natural functions. This tends to minimize the effect of the correction coefficient, ϕ, because when t is high, ϕ approaches one.

Their synthetic function's rate of growth is also distinctive in that it falls swiftly through the range of customary interest rates -- swiftly, at least, for the advanced years at which it reaches them. The combination of these two traits in the synthetic function virtually predetermines the optimum rotation, leaving only a narrow range of years within which cost factors may affect the solution. Little wonder, then, that their Faustmann solution comes very close to Allen's and Fisher's.

Looking at it from another point of view, what Guttenberg, Duerr and Fedkiw have done is select a function for which the site rent or the annual value of land, a, is unusually small relative to the other factors involved. 2/ Where a approaches zero, we

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1/ Duerr, Fedkiw, and Guttenberg, op.cit., pp. 33-37.
2/ A ϕ-value near unity corresponds to a low soil rent. See Chapter III, Section 1, d, ii, "The Interest Rate," below.

have seen in equations (4) and (14a) 1/ the Faustmann solution approaches the Allen-Fisher solution.

On page 34 of their joint publication, Duerr and the others tabulate computed optimum rotation ages from their synthetic function with interest rates ranging from 3 per cent to 6 per cent, and regeneration costs ranging from zero to $50 per acre. Four of the optimum rotations which they compute even correspond to negative values of a, in which cases the Allen solution actually yields shorter rotations than the Faustmann. In all their rotations the value of a is minuscule, and its effect equally so. Again, little wonder their numerical example yields the conclusion it does.

To suggest the limitations of the joint authors' generalization, I have selected a number of commercial species whose yield functions prescribe short rotations and which therefore allow greater Φ-effects. Table 2 presents the percentage growth rates of the synthetic function of Duerr and others, compared to growth rates of these selected species. The contrast is self-evident. By age 50 the Duerr function is growing at 8.5 per cent, whereas most of the others have fallen below 2 per cent.

At the foot of Table 2 are the optimal rotation ages, corrected and uncorrected, with the difference between solutions and the percentage increase of rotation age due to dismissing Φ. At 2 per cent, dropping Φ increases the Duerr rotation by 8 per cent but increases the Upland Oak rotation over 370 per cent; Slash Pine, 133 per cent; Loblolly Pine, 60 per cent; and so on. At 5 per cent, dropping Φ increases the Duerr rotation not at all but still increases the others by substantial percentages.

On the other hand there are growth functions which Duerr's synthetic function fits reasonably well. The last column of Table 2 presents one that represents a selective measurement of Douglas-fir on an indifferent site. And were Table 2 representative of all yield functions it would contain more such as this last one.

The issue I would take with Duerr and others is not, therefore, an absolute, but one of emphasis in practice. Under certain conditions their short cut to financial maturity is accurate enough, and I would not urge anyone to tax his patience under those conditions by applying a correction factor that does not correct.

Under what conditions, then, would one expect to find growth functions such that Φ affects optimal rotation ages importantly? The power of Φ hangs on several factors: interest rate, site,

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1/ Page 19.

harvest and regeneration costs, species, mensuration standards, stocking, and price anticipations, among others. On the next pages is presented an analysis of how these factors affect the power of ϕ. The discussion also affords an opportunity to submit some instructions and cautions on the use and misuse of ϕ, such as prudence dictates be attached to any new technique.

ii. Factors affecting the importance of ϕ

1. The Interest Rate. ϕ usually has more influence at lower interest rates, as Duerr et al. have pointed out (loc.cit.). In Table 2, ϕ has more influence at 2 per cent than at 5 per cent with only one exception. This is because at higher interest rates ϕ approaches unity.

Table 3 is abstracted from Appendix C to afford a bird's-eye view of how ρ/ϕ behaves from years 1-100. It is evident that the effect of dividing ρ by ϕ becomes less as longer time periods are used and has little effect at all for periods longer than 60 years except at unrealistically low interest rates.

In terms of soil rents one may understand this most readily by noting that

$$(g - C_o)\ \frac{\rho}{\phi} = g\rho + a \qquad \text{(6b) } 1/$$

That is, dividing ρ by ϕ is a shorthand way of finding $g\rho + a$. At higher interest rates a becomes smaller, as its definition makes clear. 2/ Hence at higher interest rates a affects rotations less, and so therefore does ϕ.

An important corollary is that omitting ϕ makes rotations unduly sensitive to changes of interest rates. This is evident from reading across the rows of Table 3. Reading to the right, as ρ becomes higher, so does ϕ, so that ρ/ϕ does not increase percentagewise by nearly as much as does ρ alone. The increase of ϕ damps the effect of the increase of ρ.

1/ Proof: $(g-C_o)\ \frac{\rho}{\phi} = \rho\left[\frac{g(1+i)^t - C_o(1+i)^t}{(1+i)^t - 1}\right] =$

$$= \rho\left[\frac{g\left[(1+i)^t - 1\right] + g - C_o(1+i)^t}{(1+i)^t - 1}\right] =$$

$$\rho\left[g + \frac{g - C_o(1+i)^t}{(1+i)^t - 1}\right] = g\rho + a$$

2/ See equation (3), p. 6, also footnote 1, this page.

Table 3. Short Table of ρ/ϕ (Corrected ρ), Five-Year Intervals of Years 1-100 a/

t	i= / ρ =	0.25% .00250	0.50% .00499	0.75% .00748	1.00% .00995	2.00% .01980	3.00% .02956	4.00% .03922	5.00% .04879	6.00% .05827	7.00% .06766	8.00% .0769[illegible]
1		100.2	100.3	100.5	100.5	101.0	101.5	102.0	102.5	102.9	103.4	103.9
5		20.2	20.3	20.4	20.5	21.0	21.5	22.0	22.5	23.1	23.6	24.1
10		10.1	10.3	10.4	10.5	11.0	11.6	12.1	12.6	13.2	13.8	14.3
15		6.8	6.9	7.1	7.2	7.7	8.3	8.8	9.4	10.0	10.6	11.2
20		5.1	5.3	5.4	5.5	6.1	6.6	7.2	7.8	8.5	9.1	9.8
25		4.1	4.3	4.4	4.5	5.1	5.7	6.3	6.9	7.6	8.3	9.0
30		3.5	3.6	3.7	3.9	4.4	5.0	5.7	6.3	7.1	7.8	8.5
35		3.0	3.1	3.3	3.4	4.0	4.6	5.3	6.0	6.7	7.5	8.3
40		2.6	2.8	2.9	3.0	3.6	4.3	5.0	5.7	6.5	7.3	8.1
45		2.4	2.5	2.6	2.8	3.4	4.0	4.7	5.5	6.3	7.1	7.9
50		2.1	2.3	2.4	2.5	3.2	3.8	4.6	5.3	6.2	7.0	7.9
55		1.9	2.1	2.2	2.4	3.0	3.7	4.4	5.2	6.1	6.9	7.8
60		1.8	1.9	2.1	2.2	2.8	3.6	4.3	5.2	6.0	6.9	7.8
65		1.7	1.8	1.9	2.1	2.7	3.5	4.3	5.1	6.0	6.8	7.8
70		1.6	1.7	1.8	2.0	2.6	3.4	4.2	5.0	5.9	6.8	7.7
75		1.5	1.6	1.7	1.9	2.6	3.3	4.1	5.0	5.9	6.8	7.7
80		1.4	1.5	1.7	1.8	2.5	3.3	4.1	5.0	5.9	6.8	7.7
85		1.3	1.4	1.6	1.7	2.4	3.2	4.1	5.0	5.9	6.8	7.7
90		1.2	1.4	1.5	1.7	2.4	3.2	4.0	4.9	5.9	6.8	7.7
95		1.2	1.3	1.5	1.6	2.3	3.1	4.0	4.9	5.8	6.8	7.7
100		1.1	1.3	1.4	1.6	2.3	3.1	4.0	4.9	5.8	6.8	7.7

a/ The writer is indebted to Sharon Jackson of the Giannini Foundation Statistical Pool for help in computing Appendix C from which this table comes.

This effect is most pronounced in the early years. Reading down the columns of Table 3, note that all start from virtually the same value at Year 1 but move down at advanced ages to values approaching the respective ρ - values. This represents the shift from the primary importance of site rent, where rotations are short, to the primary importance of interest on timber values where rotations are long.

To put it directly in terms of site rent: as ρ increases, site rent decreases, and this partially offsets the shortening effect of the higher ρ. At Year 1, for example, interest on capital is a negligibly small influence on rotations compared to site rent, so that ρ/ϕ barely increases percentagewise at all as ρ increases from 1/4 up to 8.

The practical effect of this is shown in Table 2C, which shows the change of rotation ages between 2 per cent and 5 per cent, and contrasts the changes as between corrected and uncorrected rotations. As the interest rate falls from 5 per cent to 2 per cent, the uncorrected rotations lengthen a good deal more than do the corrected ones.

2. Site. ϕ usually takes more effect on better sites. Growth gets off to a faster start on better sites; trees approach biological maturity earlier, and also become more crowded. So percentage yields fall off while stands are still fairly young, and ϕ can still have some effect. In terms of site rent this is to be expected, of course: since a is higher on better sites, a would naturally influence rotations more on better sites. An example is the second pair of European Larch rotations in Table 2, one on Site I, the best, and the other on Site V, the poorest. The Site V rotation is much longer and much less affected by ϕ. If anything, the contrast is less than typical.

The growth functions of Table 2 are mostly on better sites, because they were deliberately selected to show ϕ to good advantage. The Douglas-fir function selected to match the Duerr function is from a medium site. This is not to say that yields on medium sites usually match Duerr's function -- that depends on the standards of measurement, a question considered presently.

It is important to use ϕ on better sites, not just because it takes more effect there, but also because on better sites competition from alternative land uses is probably more keen, and forestry must put its best foot forward to justify its tenure.

3. Regeneration and Harvest Cost. For lack of adequate data these costs were omitted from Table 2. They would be subtracted from g in the denominator of the percentage yield fraction, thus increasing percentage yields, protracting rotations, and weakening the effect of ϕ. Thus Table 2 tends to overstate the importance of ϕ. The magnitude of the overstatement has not been determined. Harvest and regeneration costs are not easy to find in conjunction with particular yield tables. Future critics, especially foresters more familiar with harvest and regeneration costs, will probably want to correct for this omission.

It seems quite certain, however, that the correction would not invalidate the general results. Regeneration cost is often very low, as some forests are quite obliging about reseeding themselves with a minimum of human effort. And the best sites yield several times more lumber per acre than the poorest on which forestry is practiced, with little or no increased cost. One set of tables, for example, shows that Douglas-fir on Site V takes 160 years to yield 20 thousand board feet per acre, while on Site I it takes only 35 years. (McKeever, 1947). Site I thus yields nearly five times more annual growth, which would make it many times more productive when we consider the effects of interest, and relatively fixed regeneration costs. The difference implies that Site I must yield a considerable economic rent.

No doubt, too, harvest costs per acre increase somewhat with the volume of timber per acre. They would increase with age, therefore, and not increase the rotation as much as if they were constant with time.

4. Species and Use. Some species grow faster than others, and these usually let ϕ take more effect. White and Loblolly Pine, for example, grown for pulp or fuel 1/ are harvestable as early as 15 years of age (*Farm Forestry*, 1956). Redwood, despite its famed longevity, is a fast-starting species that yields more wood per acre than almost any other in its first 20 years, and whose percentage yields fall off quite young (Hiley and Lehtpere; Bruce, 1923). Red Alder is a fast-aging species that is senile by 80 (Johnson *et al.*, 1926, p.36). Such products as bamboo, Christmas trees, and nursery stock bring us down to extremely short rotations, where interest on growing stock is quite eclipsed by site rent.

The shape of the percentage growth curve is also important. 2/ If percentage yields drop off very quickly just before reaching ρ, then ϕ has little effect, even if site rent is high and ρ/ϕ is considerably higher than ρ alone. This is the case with the Eastern Cottonwood function of Table 2.

In a few instances, percentage yields may drop off much steeper just before 2 per cent than before 5 per cent (or any other pair of lower and higher interest rates) to produce the anomaly of our Massachusetts Red Oak function, for which ϕ takes more effect at 5 per cent than at 2 per cent.

1/ In South Carolina, it has been estimated that one-third of all timber cut is used for fuel.

2/ To the mathematician, that is the second derivative of the logarithm of the growth curve.

For biological resources other than timber, ϕ would probably take more effect, due to the much shorter rotations. Alcoholic beverages, for example, mature for market under five years -- even that figure gives the distillers and vintners the benefit of the doubt; cattle under three; oysters under two; honey under one; and so on.

The increased effect of ϕ would be partly offset, in these instances, by much higher explicit costs of several kinds, which we could never treat so incidentally as with timber. ϕ-effect might even be less with some products. The effect at any rate would probably repay study by agricultural economists.

For short rotations such as these one requires an answer more precise than to the nearest year. In this case probably the best procedure is to use months as units. This entails using much lower interest rates, dividing the annual rate by 12. For this purpose I have included rates below 2 per cent in the tables of ρ/ϕ in Appendix C. Where division by 12 gives a rate not in the table, one may use any fraction of the year as a unit. For example, 5 per cent annually could be converted to 1/2 of 1 per cent every tenth of a year, or 1/4 of 1 per cent every twentieth of a year.

5. Standards of Forest Mensuration. Yield functions and optimal rotation ages change according to what one is trying to get from a forest, and hence what one defines and measures as the "yield. The difference in reported "yields" by different measurement standards is huge, as became evident after World War II when logged-over, burned-over timberlands, thoroughly "depleted" by prewar standards, yielded large additional supplies. In general, more intensive forest utilization leads to shorter rotations, while extensive or "cream-skimming" forestry brings longer ones. This is set forth in Table 4, which shows optimal rotation ages for several species at different standards of measurement.

The cubic foot, cordwood, or other volume measure represents the more intensive standard. It is appropriate where wood is grown for pulp, or fuel, and also where milling technique has developed, as in some European countries, to the extent that most of the wood volume can be used. The board foot measure, on the other hand, represents an emphasis on large sawlogs. It excludes trees below specified diameters and allows for large mill wastes. The "Scribner rule" is the most selective, and the "International rule" somewhat less so. These measures are more common in our Northwest lumber region.

The monetary measure may represent greater or lesser intensity than the board foot, depending on what is valued at the time and place. Some monetary tables will put a higher value on larger sawlogs and peelers, leading to even longer rotations. On the other hand, European monetary yield tables often lead to shorter rotations than American board foot tables because European

Table 4. Optimal Rotation Ages, Uncorrected and Corrected, with Yields Measured in Volume, Board Feet, and Money, at 2 Per Cent and 5 Per Cent

Species and Measurement	Optimal Rotation Ages							
	At 2 Per Cent				At 3 Per Cent			
	Uncor-rected	Cor-rected	Diff.	% increase	Uncor-rected	Cor-rected	Diff.	% increase
	(years)				(years)			
Loblolly Pine, III a/								
Cu. Ft.	44	38	6	16	35	32	3	9
Dollars	52	45	7	16	43	42	1	2
European Larch, II b/								
Cu. Ft.	44	33	11	33	31	27	4	15
Pounds, Sterling	65	45	20	44	38	35	3	9
Yellow Poplar, Site 120 c/								
Cu. Ft.	50	30	20	67	28	22	6	27
Bd. Ft.	55	44	11	25	35	31	4	13
Douglas-fir								
Cu. Ft. (Eng.) d/ (thinned) (U.S. Site III)					35	30	5	17
Cu. Ft. (Denmark) e/ (thinned)					45	39	6	15
Bd. Ft. (U.S.) f/ Site I	64	54	10	19	44	42	2	5
Redwood II g/								
Cu. Ft.	45	29	16	55	27	20	7	35
Bd. Ft.	54	45	9	20	38	34	4	12
Ponderosa Pine h/								
Cu. Ft.	48	20	28	140	23	20	3	15
Cu. Ft. (trees 11.6" d.b.h. and larger)	55	41	14	34	34	30	4	13
Bd. Ft.	62	45	17	38	36	31	5	16
Norwegian Spruce i/								
Volume	61	50	11	22	42	38	4	10
Crowns	80	67	13	19	49	47	2	4
Douglas-fir Site 140' (III)								
Bd. Ft., Comprehensive j/	51	38	13	34	34	31	3	10
Bd. Ft., Choice saw logs only k/	102	95	7	7	73	73	0	0

Table 4. (continued)

Sources of primary yield data used to compute data in Table 4.

a/ Davis, 1954, p. 255.
b/ Hiley, 1930, p. 127.
c/ Forbes, 1955, p. 43.
d/ Barnes, 1955, 1956.
e/ Management of Second Growth Forests in the Douglas-fir Region, p. 11.
f/ McArdle, 1930, p. 27.
g/ Forbes, 1955, p. 44, cited from Bruce, 1923.
h/ Forbes, 1955, pp, 28-32, cited from W. H. Meyer, 1938.
i/ Petrini, 1953 transl., p. 129.
j/ Forbes, p. 3., cited from Schumacher, F.X., 1930. "Entire stem, including stump and tip, but without limbs or bark."
k/ McArdle, 1930, p. 67. "Trees 15.6" in diameter and larger, to a 12" top, Scribner rule. Trees scaled by 32-foot logs. Allowance was made for a 2-foot stump."

mills are more adapted to smaller logs, and the monetary table will give weight to small growth volume that escapes the American board foot cruiser. Figure 1, for example, is based on a European monetary yield table on English Site II, which is equivalent to our Site III, and still it gives a rotation short enough for a considerable ϕ-effect.

Monetary tables usually represent a more selective standard than volume tables. They also deduct harvest cost, the residual being called "stumpage." This deduction also tends toward longer rotations and lesser ϕ -effects, except where per acre harvest costs increase appreciably with growth.

One may inquire why any but stumpage or monetary tables should be used for economic analysis. The answer is that not many are available. Relative prices of logs vary so much with time and place that foresters have concentrated their work on physical measures of more general usefulness. Development of monetary yield tables tailored to regional market structures lies largely in the future, and would make a major contribution to rational forest management.

Table 4 makes it evident that more intensive measurement standards lead to shorter rotations. Volume develops earlier than "quality" where quality is conceived in terms of large logs. So percentage yields drop off earlier by the volume measure.

The shorter volumetric rotations also usually show greater ϕ-effect because ϕ is farther from unity at earlier ages. This is partly offset by the fact that quality increment comes just as volume increment is declining, so that quality-measured functions drop more slowly through the range of customary interest rates, allowing more scope for ϕ to affect rotation ages. In a few instances this latter effect prevails.

In general, intensive standards of forest mensuration correspond to dear timber, which in turn corresponds to scarcity of timber land, and to high site rents. It is to be expected, therefore, that intensive standards of mensuration should correspond to large ϕ-effects.

6. Stocking. Yield tables and optimal rotations change according to the degree of stocking of a site. The yield tables presented in Table 2 represent "full stocking," a somewhat fuzzy concept based on fullness of the forest canopy, or on the happy assumption that growth is proceeding without much damage from fire, insect, or blight. These are called "normal" yields but are normal in about the sense that par golf is normal. Most stands are "understocked" and grow along different paths from the yield functions of Table 2. It is a moot question whether "normal" tables represent economical stocking, or an unrealistic idealized professional standard. Probably there are instances of each, and we should consider both possibilities.

(a). Understocking economical. Suppose, first, that the prevailing understocking is economical. Then we need only find the yield function for the understocked stand and apply ϕ just as before. I have done so in Table 5 for two species and have compared the resulting rotations and ϕ-effects with those for fully stocked stands. In both instances understocking leads to greater ϕ-effects.

These two examples represent several others from the same two sources. But still the number is too small to draw any but tentative conclusions. Generalization is the more hazardous because the increased ϕ-effect comes about in different ways depending on whether one uses a volumetric or a monetary measurement.

The Loblolly Pine table is volumetric. Here we see the "trend to normality" of understocked stands. Understocked stands produce less wood per acre in their early years. Although individual trees grow faster, there are fewer of them. Later on, however, when growth of fully stocked stands is choking off, understocked stands still have Lebensraum for spreading out. Faster growth on a smaller base during this period sustains their percentage growth rates so that these rates fall very slowly through the range of conventional interest rates. This allows extensive scope for ϕ to affect the rotation age.

On the other hand, of course, lengthening the rotation tends to weaken ϕ by driving it toward unity. In the example given, this latter influence succumbs to the former, so understocking does strengthen the ϕ-effect, on balance. But a priori either influence might prevail. Only from extensive empirical studies could one generalize. This would make a fruitful topic for future research.

Table 5. Effects of Understocking on Growth Paths and Rotation Ages

	Douglas-fir a/ S.200'		Loblolly Pine b/Site 80'	
	Stumpage ($)		Cubic Feet	
	Heavy Stocking	Low Stocking	100% Stocked	20% Stocked
	(per cent)		(per cent)	
20				
25			10.0	10.7
30		11.6	5.19	6.7
35	13.7	8.26	3.03	4.53
40	8.0	5.55	1.62	3.31
45	5.65	4.16		2.67
50	4.78	3.48		2.07
55	4.11	2.99		
60	3.11	2.44		
65	2.46	2.06		
70	1.76	1.73		
Solutions:				
At 2 Per Cent				
Uncorrected	68	65	39	51
Corrected	62	55	32	38
Difference	6	10	7	13
% Increase	10	18	22	34
At 5 Per Cent				
Uncorrected	49	42	30	34
Corrected	46	40	29	31
Difference	3	2	1	3
% Increase	7	5	3	10

a/ Grah, 1957, Table 48.
b/ MacKinney and Chaiken, 1939, Table 12 and p. 28.

The Douglas-fir table presents a contrasting picture. It is monetary, based on quite a selective measurement emphasizing high grade sawlogs and peelers. Here the increased ϕ-effect comes about in the opposite way, through a shorter rotation.

Rudolf Grah, who provided the basic data, has concluded that the trend to normality loses its force where Douglas-fir is measured by high quality standards (Grah, 1957, p. 175). This is largely on the principle "as the twig is bent grows the tree." Understocking produces inferior trees: their faster individual growth makes for low-quality wood, and their wide spacing lets lower limbs develop which produce large knots and put a larger share of the volume in nonusable form (cf. McArdle, 1930, Plate 7).

It is also worth noting that uneven spacing weakens natural selection because the accident of favorable location, e.g. at the edge of a grove, lets many weaker trees outcompete stronger trees in poor locations, e.g. in the center of a grove. Uneven spacing also leaves open spaces which may grow up to weed trees.

So once off to a bad start, an understocked stand continues to lay on low-quality wood. Grah's monetary yield tables actually produce shorter rotations for less fully stocked stands.

His less fully stocked stands' percentage yields also fall fairly slowly through the range of conventional interest rates, and so evince greater ϕ-effects than his more fully-stocked stands. How general this is remains for future investigators to determine. Grah is one of the first to supply usable data on this subject. yod!

(b). Understocking not economical. Let us next consider the second possibility, that understocking is not economical. In view of the many defects of understocked stands, this is probably often the case. And even were say 50 per cent stocking the most economical standard, there is plenty of timberland less fully stocked than that. It is on these uneconomically understocked stands that ϕ is most effective. But it is also here that there is greatest danger of misusing it.

Uneconomical understocking poses a new analytical problem. We cannot assume now, as hitherto we have, that the present rotation is to be repeated after its harvest. The present rotation yields less than the optimal site rent. We must therefore figure the potential site rent from the next rotation. It is that future site rent that represents the annual gain foregone by keeping the site under the present stand. 1/

1/ For fuller treatment of this point see Chapter IV, Section 3, below.

It would be a serious error, therefore, simply to apply the ϕ-correction to the yield table of an uneconomically understocked stand to find the rotation age. Rather, one should apply ϕ to the yield table for the next rotation and compute the site rent (a) for that, using the best forecasts and most economical methods available at the time. 1/ Then apply this future a in Equation (4), $g' = g\rho + a$, where g and g' come from the present rotation. It is economical to let trees stand only so long as their annual growth, g', covers both interest on their stumpage value, $g\rho$, and the optimal future site rent, a.

This procedure will hasten the harvest of uneconomically understocked stands and their conversion to economical stocking. This is all to the good for it is folly to hold productive sites under puny stands simply because the stands are earning good returns on their own meager values. Unless they are also earning a market return on the site value, i.e., covering the site rent, a, it is well to clip their tenure short and release the site for more productive use.

In this situation we must concede that the use of ϕ entails more extended computations than does the simple Allen-Fisher solution. But it is here that its use is most advantageous. One of forestry's most pressing practical problems is to convert neglected understocked lands to vigorous progressive management. Explicit recognition of potential site rent from future timber crops is a most effective means to this end.

7. Probability of Physical Damage. The growth schedules shown in the preceding tables are based on the assumption that growth proceeds unhindered by fire, blight, or insects. But in fact, in 1952 mortality from these causes equalled about 20 per cent of the net growth of the nation's timber. 2/ Ideally, one should compute the probability of loss of a given amount, reduce this to a unitary annual value, and deduct it from g' in each year. This would tend to shorten rotations.

1/ Having determined the length of the future rotation, it is easiest to compute a from this formula:

$$a = \frac{\rho}{\phi}(g - C_o) - g\rho.$$

The proof of the validity of this relationship is found on page 33, note 1.

2/ Statistical Abstract of the United States, 1957, pp. 693, 696. Cited from the Timber Resources Review.

To be sure the expectation of losses in future rotations would also lessen the site rent, a, offsetting the decline of g' in part. And if the probability of loss were a constant annual charge, in fact, it would not affect rotations at all. 1/ But it would surely increase with time. Not only are older trees worth more, but also more susceptible to insects and disease. The probability of loss therefore increases with age, and would tend to shorten rotations. Working out actual annual charges for the probability of loss would make an important contribution to economical forest practice. Until this is done, we cannot say how such an allowance would influence the effect of ϕ.

8. Intermediate Costs and Revenues and Anticipated Price Increments. These two factors which sometimes affect the importance of ϕ are treated later. 2/

To summarize: the forester may safely neglect ϕ when and where interest rates are very high; land is marginal; harvest and regeneration costs are very high; species grow slowly and mature abruptly; the market demands top grade sawlogs and peelers only; and stocking is full. Under the opposite conditions he will often find it pays well to take the trouble of correcting his rotations with ϕ to economize on the forest site.

The long run trend seems toward those conditions, making ϕ important. The United States is rapidly emerging from an exploitive, land-rich, capital-poor frontier economy to an importer of raw materials -- an importer that needs to husband its scarce timberlands with increasing care. Concurrent technological changes should also increase the importance of ϕ. Logging and regeneration costs tend to fall as machinery and technique improve.

Timber will come to mature faster through several forces: diffusion of better species and better adaptation of species to site; better forest management; and research in forest genetics, fertilization, and endocrinology, which hold forth some tantalizing possibilities for speeding growth.

Mills now geared to handle large virgin logs will have to adapt to smaller ones as the virgin timber disappears. This should increase the relative value of smaller logs because a key to economical milling is uniformity of log sizes (Hiley, 1955). Second-growth forests produce large quantities of small logs, even when managed for large logs -- Shirley remarks that the forester who overlooks these loses half the output of his land (Shirley, pp. 266-7). In a second-growth forest economy large-log equipment could become the expensive extra, since small logs must be handled in any event.

1/ See Chapter IV, Section 1, below.
2/ See Chapter IV, Section 1 and Chapter V, Section 1.

In a dynamic progressive economy, it is likely, too, that many stands will become as a general rule partially obsolete before they reach maturity. New knowledge gained in the years since planting will prescribe a better future planting, yielding higher soil rent. In this environment a simple application of ϕ would prescribe too long a rotation, as it would implicitly understate site rent, a. Obsolete stands should be analyzed like understocked stands, borrowing a's value from the best possible future rotation.

In the light of the present recession, the words "long run" bear emphasis in the above conjectures. No industry is more cyclical than lumber's biggest customer, construction, and cylical variation may quite obscure long-run trends for a number of years.

2. Boulding's Maximum "Internal Rate of Return"

Kenneth Boulding has advanced yet another solution.1/ He rejects Allen's and Fisher's solution on the solid ground that it fails to take account of future rotations. Boulding does take account of them, and concludes the solution is to maximize what he calls the "internal rate of return," meaning the annual rate at which the original investment, C_o, would have to grow to equal the stumpage value, g, at harvest time:

$$i_B = \left(\frac{g}{C_o}\right)^{\frac{1}{t}} - 1 \qquad (16)$$

The age of maximum i_B is the same, writes Boulding, as the age of maximum economic rent. Indeed, his proof consists largely of an attempt to demonstrate the identity. He never refers to Faustmann's method of maximizing site rent, nor the discrepancy

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1/ Boulding, 1955, Chapter 39. The proposition is also contained in the 1948 edition of Boulding's Economic Analysis, but differently supported. The present remarks are addressed to the 1955 edition.

Hildreth (1946, p. 158) and Scitovsky (1955) have also advanced this solution for limited special circumstances. Worrell (1953) and Redman (1956, p. 906) cite Boulding approvingly, although Worrell's approval is based on Boulding's maximizing economic rent, which we will see he does not actually accomplish. Redman, curiously, seems to be citing Boulding in support of Allen's and Fisher's solution.

of the solutions. 1/

Boulding's proposition is of interest not only as a concept of financial maturity, but for two startling by-products. Boulding alleges that marginalist reasoning, applied to time economics, leads to error; and that changes of market interest rates, however great, should not affect rotation ages one iota.

Before systematically refuting Boulding's demonstration, it is instructive to note how his solution breaks down at one extreme. Suppose regeneration cost, C_o, approaches zero, as in fact it does on sites where natural regeneration occurs. Boulding's "internal rate of return" approaches infinity — a pleasant but not a plausible outcome.

A timber management consultant using Boulding's solution would advise his clients to limit their planting to self-stocking sites, on which sites minute investments would yield nearly infinite percentage returns. But he would not let them pay a cent for the sites themselves, since Boulding's method allows no return to the site whatsoever. He would have to insist, in fact, that they sell whatever land commanded any market price. As they converted to this system, they would make higher and higher percentage gains on less and less land until reduced to an extra terrestrial figment in an economic Nirvana of infinite internal rates of return. This outcome would spare them many more headaches implicit in a system of dual interest rates: nothing on the site value, and everything on the regeneration cost.

Other costs. Harder

What Boulding has done is to impute all returns above C_o to one input leaving nothing for the others. Therefore instead of "optimizing," finding the best combination of all inputs, he maximizes the return to one alone. Allen's and Fisher's error was similar, but where Allen simply left the soil rent unimputed, Boulding imputes it, and all the accumulated compound interest on it over the years, to interest on the planting costs, C_o.

1/ Boulding actually takes wine, not timber, as his example, but his treatment is quite general in respect to technology. He advances his conclusion as a general one, and some foresters have cited it as bearing on timber rotations. "Site rent" in this case would be the return on storage space in the wine vats. In practice this would differ from forestry in that wine vats depreciate, but Boulding does not introduce this factor at all.

How did Boulding undertake to demonstrate a proposition with such unlikely implications? He assumes that it is desirable to maximize annual economic rent, a premise with which we cannot quarrel, and tries to show that the rotation age maximizing his "internal rate of return" also maximizes the economic rent. "If therefore," he writes, "the maximum rent is charged, the investor is forced to adopt that period of investment at which the internal rate of return is maximized." 1/ In terms of our Figure 1, Boulding alleges that B is simultaneous with F.

Boulding's support of his proposition, like that of Guttenberg *et al.* consists of one numerical example. But one does not prove general propositions with unique numerical examples. 2/ Worse, Boulding's function is discontinuous, with very large gaps between the defined values. Still worse, even in the numerical example offered, the Faustmann and Boulding solutions are not simultaneous, as alleged. We will demonstrate that in general they never can be, except when soil rent is zero.

Table 6 sets forth Boulding's numerical example. By his reckoning soil rent, a, and the "internal rate of return," i_B, are maximized simultaneously at year 2. He does not specify the time of year, but presumably means January 1. He does not consider the possibility of maxima occurring at other times of the year.

We have already demonstrated that soil rent, a, is a maximum when,

$$t_F = \frac{1}{\rho}\ln \frac{g'}{g' - \rho(g - C_o)} \qquad (6) \quad 3/$$

1/ Kenneth Boulding, Economic Analysis (3d. ed., New York: Harper & Bros., 1955), 371.

2/ This may be an occupational hazard. A third instance is that of R. S. Kearns, who undertook to demonstrate the identity of the Waldrente and Bodenrente solutions using two numerical examples. For this the forester, Roy Thomson, has duly chastened him. Roy B. Thomson, "Are Similar Results Always Obtained by Use of Soil Rent and Forest Rent Procedures?" Journal of Forestry, 38 (1940), 792-793. In all three instances the effort was to identify with the Faustmann solution, whose critics do thus accord it a certain invidious esteem.

3/ Chapter II. Several other proofs are in Appendix A.

Table 6. Boulding's Numerical Example a/

t	Total Revenue	Total Cost $C(1+i)^t$ (i = .10)	Net Revenue	Net Revenue per Yr. (simple avg.)	Annual Equivalent of Net Revenue (Computed) b/	"Internal Rate of Return"
			(dollars)			(per cent)
0	810.0	1000.0				
1	1110.0	1100.0	10	10.0	9.53	11.0
2	1320.0	1210.0	110	55.0	49.9	14.8
3	1481.0	1331.0	150	50.0	43.2	14.0
4	1629.1	1464.1	165	41.2	33.9	12.8
5	1775.5	1610.5	165	33.0	25.8	12.2
6	1916.6	1771.6	145	24.2	17.9	11.5

a/ Boulding, op.cit. Tables 77 and 78, pp. 863, 871.
b/ Received continuously.

This is time F on Figure 1. By similar techniques one may establish that Boulding's "internal rate of return," i_B, is a maximum when

$$t_B = \frac{g}{g'} \ln \frac{g}{C_o} \qquad (17)$$

This is most quickly done by setting the time-derivative of equation (16) equal to zero and solving for t. This gives time B on Figure 1. Clearly the two solutions are not in general the same. In a moment we will show that F is always greater, so long as soil rent is positive. But first let us apply these solutions to Boulding's numbers.

Boulding's discontinuous function does not tell us the exact value of g' at 2. Rather than estimate it roughly, let us calculate precisely what it would have to be for i_B to be a maximum at 2, as Boulding alleges. This we do by substituting 2 for t_B in equation (17), and solving for g'. It turns out to be 183.

$$g' = \frac{1320}{2} \ln 1.320 = 183$$

Now let us substitute 183 for g' in equation (6) and see if t equals 2.

$$2 \overset{?}{=} \frac{1}{.0953} \ln \frac{183}{183 - .0953 \times 320} = 1.915$$

It does not. This means that the Faustmann equilibrium condition is not satisfied simultaneously with the Boulding condition. 1/

This conclusion is confirmed and generalized by close inspection of lines β, T, and δ on Figure 1, radiating from C_0 on the ordinate. On these semi-log coordinates a curve such as δ with the ordinate $C_0(1+i)^t$, whose annual rate of growth is a constant, i, is a straight line. The maximum i_B is found by rotating upwards the straight line $C_0(1+i)^t$ -- algebraically this means increasing the value of i -- until it just touches the growth function g. This tangent is line β. It represents the highest possible value for Boulding's "internal rate of return," i_B. The t coordinate of the tangency is Boulding's solution, B.

The maximum site rent, on the other hand, is found not by rotating curve δ but by adding to it another curve showing the cumulated and compounded annual site rent, $a\frac{(1+i)^t - 1}{\rho}$. Here i (with ρ, its alter ego) is held constant and the variable to be maximized is site rent, a. The highest such curve possible without overshooting g is T. Its tangency with g has the t-coordinate F, the Faustmann solution, where site rent, a, is a maximum.

On Figure 1, F is shown to the right of B. This is a necessary relationship so long as site rent, a, exceeds zero. On the semi-log coordinates the function T is not a straight line, for its percentage rate of growth, T'/T, declines with t.

$$\frac{T'}{T} = \frac{C_0\rho + a_F}{C_0 + \frac{a_F}{\rho}\left[1 - \frac{1}{(1+i)^t}\right]} \tag{18}$$

where a_F is the value of a at F, a parameter of this curve. The curve falls over to the right as it grows, approaching the slope of δ as t approaches infinity. It is geometrically evident therefore that T must touch g to the right of B (so long as site rent, a, is positive).

Evaluation of Boulding's contention is a little more complicated by his having chosen a wrong way of annualizing economic rent. 2/ The correct way, we have seen, is to find that amount, a, which, received annually, cumulated, and compounded at interest, will grow to equal the periodic net economic rent at the end of t years (equations (1), (1a), and (3). Boulding instead takes a

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1/ The Faustmann condition is satisfied when t is slightly above 2. One's first impression may be the opposite, that t_F equals 1.915. But recall that the right side is also a function of t. The right side grows more than proportionately to increases of t, so equality is found by increasing t.

2/ Boulding, op.cit., pp. 875-878.

simple annual average of that periodic net rent -- a procedure which is consistent with his implicit assumption that no interest be charged on the site value. Call this concept J.

$$J = g - \frac{C_o(1+i)^t}{t} \qquad (19)$$

Setting the derivative of J with respect to t equal to zero and solving for t we find,

$$t_J = \frac{1}{\rho} \ln \frac{g - tg'}{C_o\ (1-t\rho\ )} \qquad (20)$$

In general this is not the same as equation (17). Again taking g' as 183 when t equals 2, this equation is not satisfied.

$$2 \neq \frac{1}{.0953} \ln \frac{1320 - 2 \times 183}{1000\ (1 - 2 \times .0953)} = 1.73$$

It is even farther from it than the correct soil rent formulation.

One might protest that we are using a sharper pencil than the accuracy of field data would generally warrant, and that in practice B affords a satisfactory approximation to F. But this, like the Güttenberg approximation, depends on the size of the soil rent. If rent is small, as it is in Boulding's numerical example, the two are close. But rent may be large, and the two very different. Since we cannot be certain how large rent is until we find F, and since it is as easy to find F as B, there seems no reason for not proceeding directly to it. Values of B and F for a number of natural growth functions are given in Table 7. Evidently B and F may diverge significantly, so long as C_o and i are low enough so that site rent, a, is appreciable.

There is more to Boulding's demonstration, but the rest falls with its key proposition that B is identical with F. But the implications for marginal analysis do warrant separate comment. Boulding rejects a marginal solution in favor of his maximum "internal rate of return" and intimates clearly that marginal techniques, applied to time economics, lead to error. Boulding subsequently minimizes the implications of his conclusion 1/, but does not retract his basic thesis. The substantive work stands as a negation of marginalist reasoning.

It challenges not only marginalism but also the related concept of optimizing, of balancing costs against revenues to maximize a net residue of economic gain. Boulding rejects a "net" solution

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1/ Boulding, op.cit., pp. 875-878.

Table 7. Boulding and Faustmann Rotations for Several Standard Yield Functions*

Item	Species and Site				
	European Larch, II, in £ a/	Douglas-fir, I, in Cu.Ft. Corrected by a Quality Index b/		Loblolly Pine, III, in Dollars c/	
Assumed C_o per acre	10	1000 units	100 units	$100	$10
Assumed i	2%	5%	5%	2%	2%
Faustmann rotation	48	32	29	48	46
Boulding rotation	33	28	21	43	35

* Data are subject to minor ambiguities of estimate due to estimating instantaneous values of g' from discontinuous data and minor error from use of slide rule and graphic interpolation.

a/ Wilfred E. Hiley, The Economics of Forestry (Oxford: The Clarendon Press, 1930), 127.

b/ Basic figures in cu.ft. from Ibid., p.243. Corrected for quality increment with an index of quality derived from Ibid., p. 127, for European Larch. Actual Douglas Fir quality indices are not available to the writer.

c/ Kenneth P. Davis, American Forest Management (New York: McGraw-Hill Book Co., 1954), 235.

in favor of imputing all revenues to a single input and maximizing returns to that one input. An analogous procedure in elementary economics would be to take maximum output per man as the optimum combination of labor with land and capital -- a perennial fallacy, with profound policy implications in many industries, whose demonstrated capacity to work mischief is our reason for dwelling on this point.

The fault, however, does not lie with marginalism. The marginalist solution which Boulding expounds and then rejects is amiss simply because the marginalist omits part of the marginal cost of time. He does correctly identify the marginal product of time as the time-rate of growth, g'. But his marginal cost of time is simply interest on the stumpage, g_ρ.1/ This yields Allen's solution, which he rightly rejects. 2/ But had he

1/ Ibid., 864-867.
2/ Ibid., 868.

included site rent, a, as part of the marginal cost of time, he would have arrived at the Faustmann solution and marginalism would have been vindicated.

3. The Choice of What to Maximize

We now have before us three distinct concepts of financial maturity. While we are to consider still others, these three have the most general interest for and support of economists. So this is a good time to pause in our catalogue of concepts to discuss the choice among these basic ones. We will see that it is sometimes valid to maximize the internal rate of return or the discounted net yield, provided one first carefully defines these as residuals net of site rent.

minimize

(a). Maximizing C_0 or i within the framework of Faustmann's formula. A few readers of this manuscript have remarked that Faustmann's solution, maximizing site rent, seems appropriate to some circumstances, Boulding's to others, Allen's and Fisher's to still others. Tibor Scitovsky probably finds the nub of this thought when he states that one should maximize the return to whichever input is limited to the firm. 1/ This contains an important element of truth, but it is a half-truth and thereby doubly mischievous. For as a rule several inputs are limited in the sense that they command a price. Economic problems would not be very interesting if only one input were scarce. The hard problems arise in striking an optimum balance among several scarce inputs.

The clear and unexceptionable superiority of Faustmann's concept of financial maturity is not his choosing the return of land, instead of another input, to maximize. Rather it is his acknowledging the joint contribution of other inputs, and allotting them their market rates of return before maximizing the residual return imputed to the site. Faustmann is simply more comprehensive than Boulding, Fisher et al. These err in that they fail to deal with one input whose presence is implicit in the problem.

Fisher's and Allen's error is not in failing to choose site rent as their residual imputee. They simply overlook it altogether. Had they allowed an adequate return to the site and then maximized net yields, defined net of site rent as well as regeneration cost, our quarrel would be reduced to minor matters of practical judgment. Likewise Boulding's error is not to maximize i, but to do it by arrogating the joint product of two inputs to the account of one alone.

Other readers have protested "Need we bother with Faustmann's troublesome expression for 'accumulating and compounding soil rent'? Instead of expressing the site's claim on the product separately as an annual charge, why not treat the site entirely parallel to regeneration cost, as an input at time t_0? Measure it by its capitalized value, V, and compute an interest return

1/ Scitovsky, op.cit., p. 373. Cf. Hildreth, op.cit., p.163.

on it. Thus you can omit the annual charge for rent, a, and simplify and familiarize the mathematics." Too, could we not be fairer to Boulding, Fisher et al. and assume they intended to subsume V as a part of C_o? Are we not perhaps just being professionally self-conscious land economists to insist that land have a separate term?

The answer is that the land input has a separate quality that demands separate treatment, even at the level of abstraction maintained by Boulding, Fisher and others, who rarely mention specific inputs.

True, it is not essential to express the site's return as a separate annual charge, a. We could assign it a capitalized value, V, lump it with C_o, and express the annual charge as interest computed on the base V_o. But this would call for one amendment to our equation. We would then have to allow for the unexhausted value of the site at harvest time.

For C_o is embodied in the product, severed with it from the site, and carried off to market. V, the site value, remains. The site does not as a rule depreciate but remains after harvest, an asset to the firm. So we must subtract its value at harvest time from total costs -- or we could add it to output if you prefer. In either case we then get the Faustmann equation:

$$g = (C_o+V)(1+i)^t - V = C_o(1+i)^t + V[(1+i)^t - 1] =$$

$$C_o(1+i)^t - a\left[\frac{(1+i)^t - 1}{\rho}\right] \qquad (12)$$

The pesky little "minus one" in the second expression on the right side is an allowance for the permanence of the site. Formulations which omit it are incomplete.

But while one cannot escape Faustmann's equation, one need not follow him so far as to maximize a on all occasions. Equation (12) yielded us Faustmann's solution when we assigned fixed market values to C_o and i, then maximized a. We could equally well assign fixed values to a and i, then maximize C_o. That would give the maximum discounted yield, like Allen's and Fisher's solution, but net of site costs. Or we could fix a and C_o, then maximize i as Boulding does, only net of site cost. If we do the last, however, we must take care to avoid an internal contradiction -- and in the process develop yet another concept of financial maturity, one that is superior to Faustmann's under certain conditions.

One does not normally lease land for a long term operation like growing timber, and in general we must be prepared to assume that the fixed price at which land is available, if we take it as externally fixed, is the price of land titles, V. This is tied to a through the interest rate: $V = a/i$ as it is usually expressed, and $V = a/\rho$ (which is for practical purposes the same) when we assume, to permit the use of calculus, that a is received continuously rather than at each year end. When we say that a is externally determined, we must mean that we can buy or sell land titles at a fixed price, V, and a is that fixed price times an assumed interest rate, ρ. But if the interest rate is the very thing we maximize, we can hardly assume a fixed one. That would give us some absurd result such as earning 10 per cent on C_o and 3 per cent on V.

The way out of this dilemma is to adjust the algebra to the circumstances. To get rid of a, substitute V for a/ρ in Faustmann's equation, just as in the intermediate form used above (Equation 12). Hold V and C_o fixed, then maximize i, the internal rate of return on V and C_o taken together.

This amounts to joint maximization of a, as previously conceived, and i. It is different from maximizing i alone, because V cannot be treated entirely parallel to C_o as Equation (12) makes clear. In the circumstance that an enterpriser's funds are strictly limited, but he can buy and sell land titles at a known market price, this blend of Faustmann with Boulding should give a better solution than either one alone. 1/

Solving (12) for i, and setting the time-derivative of i equal to zero, gives the maximizing condition:

$$t = \frac{g+V}{g'} \ln \frac{g+V}{C_o+V} \qquad (21)$$

Like our previous maximizing equations (6, 14, 17, and 20) this one must in practice be solved by some plan of directed trial and error. I have not yet contrived any method as operable as the ϕ-technique developed for Faustmann's solution. Perhaps the easiest approach would be to solve it for its constants, C_o+V, take a few trial values, and finish by interpolation. The first approximation would be Faustmann's solution, which the ϕ-technique lets us find more easily. The others would be at a lower rotation age, 2/ since this solution is sure to be shorter than Faustmann's if C_o exceeds zero, and the more so as C_o becomes larger.

It is interesting to note that whichever parameter of Faustmann's equation we choose to maximize, the solution can be reduced to the form $g' = g\rho + a$. That does not mean the solutions

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1/ Note that this same reasoning does not invalidate the results when we fix C_o and i, then maximize a, because then the assumption is that V is unknown.

2/ Hildreth's would be another good approximation, easy to compute, and slightly below Faustmann's. See Chapter III, Section 4.

are all the same, for the values of ρ and $\underline{a}$ are different in each case. But it does confirm the consistency of all solutions with each other and with marginal analysis.

(b). Which Net Return to Maximize? Now that we have the analytical equipment for maximizing the true net return to whichever input we choose, the practical question arises "Which should we choose?" We move now from the domain of simple right and wrong to an area of judgment, where we can lay down only some general guides. The problem is simplest if the forester has access to good markets where he can buy and sell all inputs at known prices, externally determined. Then it makes no difference which return he chooses to treat as a residual and maximize. Whichever it is, though, he should regard the result as tentative and prepare to expand or contract his entire enterprise until the net return he imputes to the input he treats as the residual just equals its market price.

If all inputs but one are valued in the market, then he should maximize the net residual return to that one. Usually that one would be the forest site. Land markets are so much less definitive than others that it is a good rule, in the absence of special contrary conditions, to maximize site rent.

There are several reasons for this. Other inputs go through the market in the ordinary course of production. Regeneration inputs are largely hired or bought and, when they leave the site embodied in ripe timber, are converted into money, which gives its owner command over all the alternatives that money can buy.

But the site is not produced, nor is it sold with the product. It changes hands only when firms expand, contract, take birth and die. It remains physically intact, keeping whatever specialized qualities it has, while other assets turn over and depreciate, become money, and again new assets. Spatially the site is immobile, which limits its alternatives sharply as compared with others.

There is often a very wide gap between prices bid and asked. Such prices as are put on sites may be very misleading. Accounting conventions let book values remain obsolete for decades. Tax assessments are often as bad. Rentals, where they exist, are held far below market, as on some Federal lands, by institutional and political pressures. A strong argument for maximizing $\underline{a}$ is to obviate the abuse, which would otherwise certainly be widely practiced, of selecting one of these nominal values for $\underline{a}$ or $\underline{V}$.

Another practical advantage of selecting $\underline{a}$ to maximize is that this obviates any need for estimating other fixed annual charges. 1/

1/ See Chapter IV, Section 1, below.

Now it is true that markets for loanable funds sometimes also fail to establish an unequivocal external market price. Small firms especially feel the bite of credit rationing. While often this takes the form of a stepladder of definite interest rates, rather than an unknown rate, there are firms that simply cannot borrow more. If a credit-starved firm can buy and sell land at a known price, then by all means let it take the market $\underline{V}$ as fixed and maximize its internal net rate of return, as defined in sub-section (a) just above.

Larger firms usually have good access to credit markets at known rates of interest. They also have internal funds from profits, depreciation, and turnover, often from diversified holdings, which alternative internal investments give them a basis for pricing the money. And the largest timber holder of them all, the United States, has an exceptionally well defined borrowing rate. These large holders should almost always compute the maximum site rent.

This is on the assumption these large holders are keeping their land to produce lumber. That is not always true. Timberlands are held for the mineral rights; to maintain local prices or depress wages; to establish legal claims to water, or protect watersheds; to gain anticipated price increments; to keep unwelcome voters out of controlled counties; and many other motives ulterior to timber culture. These are of great interest, but beyond the scope of this study.

As to the third alternative, it is rare that one would want to maximize $\underline{C}_o$, inasmuch as markets for regeneration inputs -- labor, materials, equipment -- are ordinarily more closely linked with external alternatives than are land and credit markets.

Now consider a situation where two or more inputs have no externally fixed market values. Here is a line beyond which many theories of imputation do not venture. Must we now throw up our hands in despair of ever finding a rational decision?

The economy of Robinson Crusoe has long made a favorite copybook example to illustrate elementary textbook principles in simplest form. It would be ironic indeed if applied economic theory should break down in actual Robinson Crusoe economies.

An individual partly isolated from markets has himself, his preferences, his assets, and on these bases can build a perfectly rational internal economy. If he cannot adjust amounts of resources by buying and selling in the open market, he can adjust their marginal productivities to correspond to their relative scarcity to himself. If, for example, he is long on land and short on capital he would want to set a low site rent and a high interest rate such as to clear the market of his little economy. Using these values he could then work out an optimal rotation period.

Since a Robinson Crusoe can create capital in the long run, and consume it more quickly than that, his own time preference if nothing else would give him an interest rate. He would probably want to treat his land therefore as the fixed factor, and maximize site rent.

To summarize: if one wishes to maximize i, or C_o, or a, he may do so without reproach provided he first allots market returns to the other inputs. This he may do within the framework of Faustmann's formula simply by singling out different elements to maximize.

The choice of what to maximize is a matter of judgment. In general, one should maximize the return to the input that is hardest to evaluate in existing markets or other alternative uses. As a rule that is the site. But small firms, constricted by stringent credit rationing, may want to maximize i. In so doing they should beware taking a as fixed, but take V instead. This gives a new concept of financial maturity that is shorter than Faustmann's, and superior to it for firms whose capital funds are strictly limited.

4. The Alleged Convergence of Solutions in "Competitive Equilibrium."

A recurrent notion is that all concepts of financial maturity are basically the same. We have considered Boulding's effort to establish one identity, and Guttenberg's, and noted Kearns'. Now let us consider the Lutzes and Scitovsky's.

The Lutzes open their discussion of our subject by listing four possible criteria, which ultimately are shown to include the three we have considered, and remark ". . . in competitive general equilibrium, when an entrepreneur just earns the going interest rate on his investment, all four criteria amount to the same thing." (p.17) Tibor Scitovsky renders the same general opinion (pp. 369-70), as does Clifford Hildreth (p. 164).

In each case the reasoning is that any excess of yield, g, over regeneration cost compounded at the market interest rate, must be a "profit" due to some market barrier or the inertia of competitors. Allow free competition and ample time and this "profit" disappears, making all solutions one.

But this is to assume that regeneration cost, C_o, is the only input. The inescapable fact is that trees require Lebensraum on the surface of our shrinking planet, a scarce resource covered with price tags. The Lutzes and Scitovsky's evanescent "profit" is in general site rent, the result neither of market barriers nor passing imbalances but of the relative scarcity of good land.

Now in defense of the Lutzes and Scitovsky one might say that they intended to subsume site value with C_o as an input at t_o. They were reasoning on a highly abstract level and mentioned no specific inputs of any kind.

But if that was their intention, it was lost in the execution. Even at the highest level of abstraction it is still necessary to account for the unexhausted value of the site at harvest. Scitovsky specifically posits the absence of any "salvage value" (p. 372). The Lutzes deal with "infinite chains" of future rotations, in which their C_o is wholly reinvested at the start of each new link, which indicates they, too, had no "salvage value" in mind. And they shortly afterward fall into error by a route that suggests they have in mind a costless, indefinitely expansible land base. 1/

It is not perfect competition, but perfect imputation that makes the solutions converge. Perfect competition is not even necessary. As we have seen, when the enterpriser can buy and sell all inputs at externally determined prices, competitive or not, it makes no difference which return he maximizes, provided only he first allows the market return to the others. 2/

It might be added parenthetically that, in the long run, the general equilibrium solution with perfect imputation would leave only the foresters using the Faustmann method operating successfully. Only they would be able to pay all factors the returns imputable to them.

We have previously noted, too, that the solutions converge on marginal land where soil rent, a, equals zero.

5. A Solution Suggested by Clifford Hildreth: Maximum Discounted Mean Annual Growth

Clifford Hildreth has advanced an interesting variant concept of financial maturity (Hildreth, 1946, p.162). His general thesis is that the concept of maturity should vary with the technique of production, so the present concept by no means represents the full scope of his analysis. He is, if anything, probably more favorable to maximizing the internal rate of return. But the present concept is, so far as I know, unique with him.

1/ See Lutz and Lutz, pp. 33-34, discussed above in Chapter III, Section 1, b, ii.

2/ See Chapter IV, Section 3, b.

Hildreth, like Lutz, conceives of an uneven-aged stand, with continuous planting and felling. But unlike Lutz he specified a fixed site, originally bare, which he plants bit by bit just fast enough so that as he drops his spade, he seizes his axe to fell the eldest trees. Now this obviously involves wasting much of the site during the first rotation, an assumed waste which would invalidate the solution in general -- a fact of which Hildreth seems to be aware. As one might expect, the resulting optimal rotation is shorter than Faustmann's, to minimize the waste of the site during the first rotation. One might protest of it, granting its assumptions, that once the overlapping rotations are well under way there is no point in letting the now past and irrelevant starting period influence later rotations.

Hildreth's rotation is especially interesting in that, like Faustmann's, it embodies both elements of urgency we have insisted on: economy of the money tied up in timber values and economy of the site. Its fault, if we look at it as a general solution -- something of which Hildreth is not guilty, but which it is useful to consider -- is that it implicitly overstates the annual value of the site.

Hildreth maximizes an expression which, to translate to our notation (and simplify by dropping out constants which do not affect it) is

$$a_H = \frac{g}{t(1 + i)^t} \tag{22}$$

As a simplification, C_o has been set at 0.

The maximizing condition is

$$g' = g\rho + \frac{g}{t} \tag{23}$$

This is the same as Faustmann's solution, except that Faustmann's soil rent, a, is here replaced with $\frac{g}{t}$, or mean annual yield. This reveals that Hildreth's method is the equivalent of annualizing the period soil rent at zero-interest. That is, the soil rent implied in Hildreth's solution is the simple average of the harvest yield divided by the rotation age, rather than the somewhat lower value of equation (3).

This formula is of interest as an analytical curiosity; as a correct solution for the rather implausible assumptions on which it is based; and as a rough approximation to Faustmann's solution that in practice might be simpler to calculate. Especially for low interest rates and short rotations, $\frac{g}{t}$ becomes very close to $\frac{gi}{(1+i)^t-1}$.

6. Zero-Interest Solutions

The nonforester probably expects a discussion of zero-interest doctrines to call up a collection of amiable eccentrics misquoting Marx, Gesell, or St. Thomas Aquinas. But not at all: the United States Forest Service itself adheres to a zero-interest doctrine, maximization of mean annual yield; and many forestry texts and schools treat zero-interest doctrines with great respect. We will survey three variants: maximum total growth; maximum mean annual growth; and maximum mean annual net growth, or Waldrente.

(a) Maximum Total Growth (Z on Figure 1)

Maximum total growth arrives when the incremental product of time, g', falls to zero. This solution implies that the incremental cost of time must therefore be zero -- that is, it dismisses both interest and site rent.

No professional forester or economist to my knowledge openly advocates this solution. Yet it would require no very unlikely combination of existing doctrines to arrive at it. Allen's and Fisher's solution already dismisses site rent. And not so long ago some economists were confidently anticipating the "euthanasia of the rentier."

Furthermore, one finds strong undercurrents of support in popular literature, based on what it is probably fair to characterize as sheer mysticism, yet which still carry weight in determining public policy.

(b) Maximum Mean Annual Growth (W' on Figure 1)

This represents a fundamental conceptual advance over Z. It achieves economy of the time-dimension of land by specifying maximum yields per year (g/t), rather than per rotation. This time-averaging step, implicit in Faustmann's formula, is the essential lack of Allen's and Fisher's. 1/

The faults of W' are to dismiss interest and also regeneration cost, C_o. True, these omissions are compensating. Dropping C_o tends to shorten rotations; dropping i to lengthen them. But reliance on compensating errors is a treacherous practice. In general W' gives too long a rotation.

1/ See the Opening Summary and the chart entitled "Some Concepts of Financial Maturity in Relation to Time Elements of Urgency."

(c). Maximum Mean Annual Net Growth, Waldrente
(W on Figure 1)

This doctrine is known in English language forestry literature as the "forest rent" doctrine, as distinct from the "soil rent" (Bodenrente) doctrine of Faustmann. The proposal is to maximize the "forest rent," W, defined thus:

$$W = \frac{g - C_o}{t} \qquad (24)$$

W is the same as W', but with C_o subtracted from the numerator. Lacking the compensating error of W', Waldrente rotations are bound to be too long.

Analysis of schedules of Waldrente suggests that the rotation is much too long, and that incremental costs of waiting from F to W far outweigh the gains. The zero-interest forester may in the last twenty years of his rotation increase mean annual net growth only negligibly, all unmindful of snowballing interest costs. Many Waldrente maxima actually correspond to negative site rents, where C_o and i are high.

Even if one wished to accept the zero-interest assumption, risking a heavy investment in standing timber to fire, insects, and disease to achieve minute increases of mean annual growth seems unwise. In addition there is a probability of loss due to changing consumer preferences and technology. There is also a chance of gain from these causes, but the longer the rotation the less opportunity the forester has to maneuver his capital among these hazards and opportunities. As in any enterprise, every turnover of capital is an occasion to adapt it to ever-shifting parameters of cost and market. The longer the rotation, the more cumbersome and unwieldy it is.

Roy Thomson traces the origin of the Waldrente doctrine back to Rodbertus, nineteenth century German socialist. Rodbertus evidently objected to Faustmann's Ricardian distinction of site from growing stock and preferred to apply the word "rent" to the entire forest income. This, combined with the zero-interest philosophy, eventuated in Waldrente (Thomson, pp. 28 ff.).

Much is written about supposed fundamental contrasts of Waldrente and Bodenrente. So far as the writer can see, this is baseless, much of it so factitious as not to warrant serious discussion 1/ Whatever the ideological or doctrinal frictions of the original antagonists, mathematically Rodbertus' Waldrente is simply Faustmann's Bodenrente with a zero-interest rate, as may be seen by applying l'Hospital's Rule to any form of Faustmann's equation. There is no other difference.

1/ Davis, pp. 239-42; Chapman and Meyer, 165-66, 253: Chapman,76 (d). Critical Appraisal of Arguments for Zero-Interest

The above three zero-interest doctrines are worth noting for two reasons. First, as limiting cases of Allen's and Fisher's and Faustmann's formulas, with interest at zero, they abstract from interest and thus lay bare the contrast in how the two formulas handle the other variable cost of time, site rent. At zero-interest Allen's and Fisher's formula devolves into maximum tree growth, with the marginal cost of time figured at zero. Faustmann's devolves into maximum mean annual net growth, with the marginal cost of time figured at $(g - C_o)/t$, the site rent at zero-interest.

Second, these doctrines warrant note because many foresters, or others in charge of managing timberlands, take them much more seriously than one would suspect from their merits.

As to Z, maximum tree growth, there are vast stands of virgin timber still held in some of our national forests, and on some private lands as well, that are not only stagnant but deteriorating. There are several reasons for this, some of which we discuss presently. But one cannot spend much time on this subject without encountering the inchoate sentiment that there is something shameful about man's cutting a tree before it has lived out its allotted span of years. The cry of "Woodman, spare that tree!" touches a sympathetic chord in us all. The feeling is bound to find an expression, however it may be rationalized. Closely related, perhaps is the austere doctrine that man's material demands are largely vanities that should not be suffered to defile the grandeur of Nature, however eager most men may be that they should.

As to W', maximum mean annual growth, our own Forest Service endorses it as a good standard of financial maturity. The Forest Service uses this criterion in its current evaluation of forestry practices in the United States and judges on this basis the relative merits of various categories of timberland holders. 1/

Timber holders who fell their trees before age W' receive demerits for "premature" cutting. If the Forest Service shows any doubt about the standard, it is that it is too short! Those who

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1/ Timber Resource Review, Chapter IV, Part B, September, 1955, pp. 16-19, 63-73. Timber Resources for America's Future, 1958, pp. 72, 671. "Effect of felling age" is one-quarter of the Forest Service "Productivity Index - a new concept in appraising forest conditions." The Forest Service Index is also open to criticism on the ground that the age at which the last stand was felled is irrelevant to the present condition of the land. For a more comprehensive critique of the Timber Resource Review, see Zivnuska, 1956.

harvest after W' receive no demerits for postmature felling. So the result is as though the Forest Service were applying a still lengthier standard. And even this is only a compromise with evil: the Forest Service looks forward to a time when it can "raise standards" by positing even longer rotations. 1/

The ineptitude of the Forest Service's standards may be exemplified by its judgment of Southern Yellow Pine -- Longleaf, Slash, Loblolly, Shortleaf. These pines account for 30 per cent of the growth of sawtimber in the United States, but only 8 per cent of the sawtimber inventory volume. 2/ That is to say their growth is very high relative to the capital tied up in growing stock, in comparison with other species. Now to some degree this reflects characteristics of the species, but the mere fact that a species grows fast would not produce this result if management policies were to hold the trees until growth had become slow. So to a greater degree this would seem to reflect management policies. And there is no group of species whose management rates quite so low on the Forest Service scale as Southern Yellow Pine. 3/

As to W, Waldrente, the forestry literature treats it with much respect and serious consideration. As Thomson tells us, few writers have subjected it to critical analysis, and most are noncommittal as between it and site rent, Bodenrente (Thomson, 1942, p. 31).

Just why foresters should be so tolerant of doctrines that to the outsider seem so patently indefensible receives little explicit discussion in forestry literature. Hiley sets forth some interesting suggestions (Hiley, 1930, pp. 218-222), to which I would add a few sadly cynical observations which I hope will be vigorously and successfully refuted.

The forestry profession in the United States seems to have inherited from Germany an inflexible and largely biological concept of "good forestry" that transcends mere local economic conditions. Above all, interest cost is unworthy to be weighed against anything as splendid as a tree. When interest cost thwarts otherwise feasible forestry projects, it is not "good forestry" that must yield. Sven Petrini writes:

1/ Timber Resources for America's Future, 1958, p. 73. There are two compensating errors in the Forest Service's criterion: first, it omits harvest and regeneration cost; second, it uses volumetric yield data, which lead to shorter rotations than monetary data as a rule. It is doubtful, though, that these two make up for the omission of interest and the tolerance of postmature felling.

2/ Timber Resources for America's Future, 1958, p. 55.

3/ Ibid, p. 77.

> In point of fact it is by no means unusual for negative soil-values to arise when applying Faustmann's formula. This circumstance is one of the main reasons for the disrepute of the soil-rental theory. 1/

Roth puts it this way:

> . . . if the forest cannot make more than 3% . . . there is little use of introducing 5% into the formulae. 2/

And Hiley, who does not hold with this idea, writes:

> The enormously high cost of production of large timber is due to the incidence of compound interest, and in order to get over this difficulty many foresters have questioned the reality of compound interest. 3/

Interest also comes under suspicion by neo-Malthusians, with their distrust of the free market as an agent for rational conservation policy. But whatever validity such thinking may have, it hardly applies to a renewable resource like timber.

Price maintenance is also sometimes a motive. It is well known that private timber holders brought great pressure to bear on the Forest Service in the 1930's to withhold its timber to avoid "disorganizing" the market. Many foresters have expressed anxiety that Faustmann rotations might make the market "collapse under a flood of small-sized products." 4/ The benefit to consumers of market "collapse" and "disorganization" is given little weight.

1/ Sven Petrini, Elements of Forest Economics, trans. Mark L. Anderson (Edinburgh: Oliver and Boyd, 1953 [Swedish publication, 1946],) 68. Cf. W. E. Hiley, Woodland Management, p. 271, for parallel observations.

2/ Filibert Roth, Forest Valuation, Vol.2, Michigan Manual of Forestry. 2d ed. revised. George Wahr, publisher. Ann Arbor, Michigan.

3/ Hiley, Economics of Forestry, pp. 219-220.

4/ Davis, 1954, p. 242. Davis' concern is nominally with the price structure, not the over-all price level, but his statement has overtones of over-all monopolistic pricing policy without which it makes no sense and Davis' rationality makes it fair to conclude he had over-all price maintenance in mind. See also Hiley, 1930, p. 212; Schlich, 1905, p. 200; and Marquis, 1939, pp. 111-12.

Another factor, of undeniable historical importance, has been mercantilism, with militarism, nationalism, and socialism, with their strong emphasis on national self-sufficiency. Roy Thomson has traced the development of forest ideology in nineteenth century Germany in these terms (Thomson, 1942, pp. 29 ff.). Zero-interest is a subsidy to an essential national industry. The Nazis, the twentieth century embodiment of these combined doctrines, in 1934 outlawed short forest rotations, Thomson points out (op.cit., p. 31), whereupon the Waldrente zero-interest doctrine gained ground from the Bodenrente or Faustmann doctrine.

The use of zero-interest has also been advocated as a compensating error to offset other noneconomic forces working toward too-short rotations. An example of this is standard yield tables based on unrealistically heavy stocking. These tables reach their financial maturity earlier than do actual stands.

If that is so, however, it is hazardous to assume that longer rotations are desirable without actually correcting the yield tables, which is the obvious remedy to take. For Grah has recently shown that in some cases understocking leads to shorter, not longer rotations. 1/ The method of compensating error is treacherous indeed.

Zero-interest is also advocated to allow for off-site and other nonsalable forest benefits, such as watershed protection, wildlife sanctuary, and recreation. But again, without direct study of the benefits it is hazardous to assume that mature forests provide more of them than young ones, and that they do not at the same time harbor more insects, blights, snags to invite fires, and other detriments to neighboring sites. Younger forests often provide a more hospitable environment for wildlife (Dolder, 1955, p. 50).

The writer has heard a forester remark: "It is too early to think about spinning out fine theories. The public hasn't yet accepted the elementary principles we try to teach them." But if one of those "elementary principles" is zero-interest, small wonder! Can it be that small woodlot owners, humanly impatient for their money, who perversely insist on harvesting "premature" timber and earning the lowest marks on the Forest Service's rating sheets, are actually economizing more carefully on their resources than the giant corporations, and the Forest Service itself, which earn the top grades?

It is a possibility that the Forest Service is hardly in a position to refute without some agonizing reappraisal of its conceptual measuring sticks. But in fairness to the small operators it condemns so roundly, 2/ such a reappraisal is very much in order.

1/ See Chapter III, Section 1,d,ii, "Stocking", p.40.
2/ Timber Resources for America's Future, 1958, pp. 75 ff.

7. Summary: An Adequate Concept of Financial Maturity

There are two elements of urgency prompting a forester to harvest his stand: economy of the site and economy of the money tied up in the trees. The first is expressed by site rent, the second by interest on the stumpage value.

An adequate concept of financial maturity must account for both elements. The fault of Allen's and Fisher's solution is overlooking site rent, thus arriving at too long a rotation. The fault of Boulding's solution is, while remembering site rent, to forget to reckon the site as an input and thus impute the rent to another input. This process, as it happens, overstates the marginal cost of time and arrives at too short a rotation.

The fault of zero-interest doctrines is, of course, to overlook the second element of urgency, interest on stumpage value. Thus they, like Allen's and Fisher's solution, arrive at too long rotations. One might also tax them with inconsistency: site rent is a percentage return on the site value, and if this is allowed, an interest rate above zero is implicitly allowed.

Hildreth's solution does account for both elements of urgency, but overstates the first, site rent, by annualizing it in effect at zero-interest. Faustmann's formula allows for both elements of urgency, using in each case a given rate of interest appropriate to the firm's financial position.

Of these concepts of financial maturity, only Faustmann's is adequate. If one prefers to maximize the discounted yield value, he may legitimately do so within the framework of Faustmann's formula. Likewise one may maximize an internal rate of return within Faustmann's formula.

In the latter event, to avoid an absurd dual interest rate in the result, he must take not site rent but site value as externally fixed, and jointly maximize the return to site and regeneration input. This affords a new concept of financial maturity that is superior to Faustmann's in limited circumstances.

Table 8 lays out the several concepts of financial maturity. See also the chart in the Opening Summary on page ix.

Table 8. Summary of Concepts of Financial Maturity

Symbol	Author(s) Advocate(s)	Expression Maximized	Name(s)	Solution	Direction of Error	Solution for European Larch, II, from Table 1
F	Faustmann, Hiley, Thomson, *et al.*	$a = \rho \dfrac{g - C_o(1+i)^t}{(1+i)^t - 1}$	Bodenrente (Soil Rent) site rent	$t = \dfrac{1}{\rho} \ln \dfrac{g'}{g' - (g - C_o)}$ or $\rho/\phi = \dfrac{g'}{g - C_o}$	0	48
A	Allen, Fisher	$\dfrac{g}{(1+i)^t}$ or $\dfrac{g - C_o(1+i)^t}{(1+i)^t}$	Discounted yield; discounted net yield	$\rho = \dfrac{g'}{g}$	+	66
B	Boulding	$i_B = \sqrt[t]{\dfrac{g}{C_o}} - 1$	"Internal rate of return"	$t = \dfrac{g}{g'} \ln \dfrac{g}{C_o}$	-	33
Z	----	g	Growth	$g' = 0$	+	Over 100
W'	U. S. Forest Service	g/t	Mean Annual Growth	$t = g/g'$	+ (usually)	75

(Continued)

Table 8 (Continued).

Symbol	Author(s) Advocate(s)	Expression Maximized	Name(s)	Solution	Direction of Error	Solution for European Larch, II from Table 1
W	Rodbertus, Borggreve, Cstwald, <u>et al.</u>	$\frac{g - C_o}{t}$	Mean Annual net growth forest rent <u>Waldrente</u>	$t = \frac{g - C_o}{g'}$	+	80
N	(Given only as a point of reference)	$g - C_o(1+i)^t$	Growth net of compounded costs	$t = \frac{1}{\rho} \ln \frac{g'}{C_o \rho}$	+	--
—	Boulding (perhaps inadvertently)	$J = \frac{g - C_o(1+i)^t}{t}$	Mean annual growth net of compounded costs	$t = \frac{1}{\rho} \ln \frac{g - tg'}{C_o(1-t\rho)}$	+	--
--	Hildreth	$\frac{g}{t(1+i)^t}$	Discounted mean annual growth	$g' = g\rho + \frac{g}{t}$	-	--
--	This study	$i = \sqrt[t]{\frac{g+V}{C_o+V}} - 1$	Internal joint rate of return	$t = \frac{g+V}{g'} \ln \frac{g+V}{C_o+V}$	Does not apply	--

CHAPTER IV

ELABORATING, GENERALIZING, AND ADAPTING FAUSTMANN'S FORMULA

1. Intermediate Costs and Revenues

For simplicity's sake, we have assumed thus far that all explicit costs came at time zero and all revenues at harvest time. In fact there are intermediate costs and revenues such as fire protection and annual taxes, and intermediate revenues from thinnings. The Faustmann formula can accommodate these with slight modification.

The modification is important not only to the handling of timber problems in their entirety. The modification permits generalizing the formula to apply to many other problems of replacement and turnover: clearing orchards, demolishing old buildings, scrapping machinery, and in other situations where costs and revenues are not concentrated at the end points. The applications outside forestry in their aggregate are of far greater scope and practical consequence than those inside it.

To begin with, constant annual costs and revenues do not affect optimum rotations in the least since the forester cannot affect them by changing the rotation. A quick mathematical confirmation of this is afforded by adding constant annual revenues to the definition of soil rent, a, and subtracting costs from it.

$$a = \rho \frac{g - C_0(1+i)^t}{(1+i)^t - 1} + R - c \qquad (3a)$$

where R is constant annual revenue and c a constant annual cost. Setting the derivative of a equal to zero, R and c drop out, not being functions of time.

It might seem that R and c should affect the optimum rotation, through marginal analysis, since they affect the incremental cost of time. But as equation (3a) makes clear, the effect of c is exactly offset by an equal and opposite change in a, which is also part of the incremental cost of time; and R increases a by the same amount as it does the incremental product of time.

As mentioned above, 1/ this fact considerably simplifies the work of computing optimal rotation ages. In many forestry texts Faustmann's formula is presented first as a means of computing "soil expectation value," S_e, and for this purpose one has to include constant annual costs and revenues. But simply to compute optimal rotations, one may omit this step.

Variable intermediate costs and revenues do affect rotation ages. Let the symbol R_t represent the algebraic sum of revenues and costs in any year, t. Let m be the year of maturity. Then the condition of financial maturity, when stated in its most operable

1/ Chapter III, Section 1, d, i

form, becomes:

$$\frac{\rho}{\phi} = \frac{g'_m + R_m}{g_m + \sum_{o}^{m} R_t(1+i)^{-t}} \qquad (6c)$$

In this form the equation and its derivation are almost self-explanatory. Future net revenues, R_t, add to land value, $\underline{V}$:

$$V = \frac{g_m + \sum_{o}^{m} R_t(1+i)^{m-t}}{(1+i)^m - 1} \qquad (9), \text{ generalized}$$

$\underline{C_o}$ is here subsumed under $\underline{R_t}$, where t = 0. And equation (6c) simply posits that the present land use just earn a return, $\underline{g'_m} + R_m$, on the value of its growing stock, $\underline{g_m}$, plus the land value, $\underline{V}$:

- mult. 6c by φ on both sides

$$\rho = \frac{g'_m + R_m}{\dfrac{g_m + \sum_{o}^{m} R_t(1+i)^{-t}}{\phi}} = \frac{g'_m + R_m}{\dfrac{g(1+i)^m + \sum_{o}^{m} R_t\ (1+i)^{m-t}}{(1+i)^m - 1}}$$

$$= \frac{g'_m + R_m}{\underline{g_m} + g_m\ [(1+i)^m - 1] + \sum_{o}^{m} R_t(1+i)^{m-t}} = \frac{g'_m + R_m}{g_m + \dfrac{g_m + \sum_{o}^{m} R_t(1+i)^{m-t}}{(1+i)^m - 1}}$$

= V ✓

$$\rho = \frac{g'_m + R_m}{g_m + V} \qquad (6c), \text{ derived in reverse}$$

In the process of applying equation (6c) in practice, one could often avoid expressing $\underline{R_m}$ separately from g'_m. In fact it would often be impossible to express them separately, for in the year of maturity and for some time previous, it would be wasteful to thin a stand separately from the major harvest. The increased volume that would have been thinned had there been no harvest would simply be included in the harvest volume, $\underline{g_m}$.

If one simply wants to compute values for equation (6c) over a series of years, it is still possible to express $\underline{R_m}$ and g'_m together. Suppose a forest is both thinned and measured quinquennially. Then the total growth from say ages 30 to 35 is the pre-thinning measure at 35 less the post-thinning measure at 30. Cr one could take the difference of the post-thinning measures plus the thinning at 35.

The only reason for expressing R_m separately from g'_m is to be ready to handle intermediate revenues other than growth of the basic timber stock. The need for this becomes evident presently when we discuss financial maturity of nonappreciating assets, for which g'_m is zero, and whose entire excuse for being rests with the "intermediate" revenues.

Table 9 presents examples of the computation of financial maturity of thinned stands, with rotation ages at 5 per cent.

Taking thinnings from a forest tends by and large to lengthen rotation ages. To be sure it increases site rent, assuming thinning is economical, which has the opposite effect. But it increases annual growth in the years near financial maturity, and it lowers the volume of standing timber on which one must charge interest. These latter two effects would ordinarily prevail.

Lengthening rotations might seem to bring Faustmann's solution closer to Allen's and Fisher's and thus warrant by-passing the lengthy computations illustrated in Table 9. But thinning would also tend to lengthen Allen's and Fisher's solution, and quite possibly the percentage difference would become even greater.

As the laborious task of computing enough thinned rotations to generalize about this would overtax our limited resources, we leave the question to future investigators. Since thinning usually accompanies more intensive forestry practices of several kinds, including more comprehensive mensuration standards, it is hard to find primary data with which to evaluate the effect of thinning in isolation. The data of Table 9 show a considerable difference of solutions, but this is of limited application. The solutions might differ less if data were in monetary terms, with thinning costs deducted from thinning revenues.

2. A General Solution to Any Problem of Replacement or Turnover

This slight elaboration of Faustmann's formula vastly expands its scope in practice. No longer is it limited to appreciating assets like timber. Wherever stock turns over, and there is a continuing implicit overhead, the formula helps find the optimal turnover period. Indeed it is hard to see how one could find the optimal turnover period without using this formula in some guise since it is necessary to evaluate implicit overhead charges simultaneously with determining the optimal turnover period.

Where the stock has no salvage value, like old fruit trees for example, one would not count appreciation of the trees as revenue--that would be double counting, as the fruit yield is counted as revenue when it is realized. Neither would one count depreciation

Table 9. Finding Financial Maturity of Thinned Stands of Douglas-fir at 5 Per Cent

Year t	Volume g, after Thinning (cu.ft.)	Removed by Thinning R_t (cu.ft.)	1.05^{-t}	$R_t(1.05)^{-t}$	$\sum_{o}^{t} R_t(1.05)^{-t}$	Annual Growth, g': $\frac{g_t - g_s + R_t}{t-s}$ *	$g_t + \sum_{o}^{t} R_t(1.05)^{-t}$	$\frac{g'_t}{g_t + \sum_{o} R_t(1.05)^{-t}}$ (per cent)
A. England. Site Index 140' (Barnes, 1955)								
13	680	170	.530	(130) 90	220	(285)	900	31.7
16	1320	280	.458	128	348	325	1668	19.3
19	2000	350	.396	139	487	338	2487	13.6
22	2600	400	.342	137	624	334	3224	10.3
25	3170	430	.295	127	751	333	3921	8.5
28	3720	450	.255	115	866	321	4586	7.0
32	4470	480	.210	101	967	302	5437	5.55
36	5150	500	.173	86	1053	292	6203	4.71
40	5785	515	.142	73	1126	265	6911	3.83
45	6470	530	.111	59	1185	234	7655	3.06
50	7055	545	.087	47	1232	218	8287	2.63
B. Denmark, Site unspecified. (<u>Management of Second-Growth</u>, 1947, p. 11)								
26	3959	929	.281	261	261	---	4220	---
29	4273	1000	.243	243	504	459	4777	9.6
32	4830	886	.210	186	690	448	5520	8.1
36	4973	1486	.173	257	947	422	5920	7.1

(Continued)

Table 9 (Continued).

Year t	Volume g, after Thinning (cu.ft.)	Removed by Thinning R_t (cu.ft.)	1.05^{-t}	$R_t(1.05)^{-t}$	$\sum_{o}^{t} R_t(1.05)^{-t}$	Annual Growth g': $\frac{g_t - g_s + R_t}{t-s}$ *	$g_t + \sum_{o}^{t} R_t(1.05)^{-t}$	$\frac{g'}{g_t + \sum_{o}^{t} R_t(1.05)^{-t}}$ (per cent)
41	6245	929	.135	125	1072	367	7317	5.02
44	6388	829	.117	97	1169	382	7557	5.05
46.5	6288	1172	.103	121	1290	364	7578	4.81
48	5931	879	.096	84	1374	342	7305	4.68
53	6160	1386	.075	104	1478	290	7638	3.80
57	5874	1329	.062	82	1560	(230)	7434	3.09

Solutions at 5 per cent:

	A	B
Corrected	30	39
Uncorrected a/	39	51
Difference	9	12
Per cent increase	30	31

*The symbol s is used to designate the year of observation and thinning preceding the given year, t. In computing $\bar{g}'$ values I put the values computed by the formula here given between the years t and s, then found the values of the table by interpolating.

a/ "Uncorrected" in this case means dropping not only Φ but also the sum of discounted intermediate net revenues. A protagonist of Allen's and Fisher's method might prefer some other adaptation in this situation, but this seems the most straightforward.

as a cost -- regeneration expense is counted as a cost once at t_o, and as the entire cycle is evaluated as a whole, there is no need to count it again. Nor would one give any value to g_t in the denominator -- there is no point in requiring an asset to earn a return on a fictitious base.

In this case therefore equation (6c) becomes:

$$\rho/\phi = \frac{R_m}{R_t(1 + i)^{-t}} \qquad \text{(6d) } \underline{1}/$$

This is equivalent simply to:

$$R_m = a \qquad \text{(6d) restated}$$

That is, g has passed altogether out of the picture, and we only posit that the old trees earn a return on the land value.

The analysis of old buildings is exactly the same. Urban land economists teach that the time to demolish an old building is when the current net income, with no depreciation considered, falls to equal the annual value of the site alone. 2/ With urban buildings, obsolescence is rapid enough so that one would usually want to apply Faustmann's formula as in the case of understocked timber stands, borrowing, that is, the site rent value from the best possible future building. 3/

Machinery replacement presents essentially the same analytical problem, with one difference. Machinery not only occupies space, it is often a vital link in an integrated operation, and its site rent must be determined in light of the potentialities of that strategic position. The same is true of those buildings that are parts of larger integrated operations.

In these problems, where the assets depreciate, the practical importance of using Faustmann's formula, or some equivalent reasoning, is much greater than in forestry. If one fails to count site rent as a cost, he will never demolish old buildings, for example.

1/ Cf. Scitovsky, p. 373n.

2/ Ratcliff, pp. 403-05. Ratcliff's treatment is incomplete, though, because it gives no clue of how to determine the site rent, which depends on the building life, which depends again on the site rent. The contribution Faustmann's formula might make to urban land economics is to determine these two interdependent variables simultaneously.

3/ See Chapter III, Section 1, d, ii.

For their salvage value is at best zero, and on this base they always earn some income thanks to the potentialities of the site. If one then overlooks site rent, this looks like an infinite percentage return and the shrewdest investment imaginable. This is in contrast to forestry, where the capital stock on the land increases continually so that financial maturity arrives presently even if one ignores site rent.

Faustmann's formula is also useful in helping plan new structures. Failure to count site rent as a cost leads to structures too small for the potentialities of the site -- a relationship often basic to the controversy between advocates of high and low dams, for example. The percentage return to a low dam on a valuable dam site can look very high if one puts no value on the site. The market for dam sites being rather limited, it is usually impossible to value the site outside a given problem. But Faustmann's formula helps one put a value on it for any given dam size, and thus find the optimal size, which is the one maximizing imputed dam site rent.

3. Changes of Data in Mid-Rotation

Faustmann's formula has been attacked as looking backward to sunk historical costs to determine present rotations. If this were what it did -- and one could use it that way -- the criticism would be warranted.

That is not to say the criticism as usually made would be warranted. The most frequent complaint seems to be that interest on sunk costs compels rotations too short for present conditions. This implies that current regeneration costs are lower than historical; and that this calls for longer rotations.

Now current regeneration costs may indeed be lower than historical costs in real terms, or relative to stumpage prices; but from equation (6a) it is clear that lower costs call for shorter, not longer, rotations. This is because lower costs mean higher soil rents, hence higher incremental costs of time. Still, it would be valid to criticize the formula for prescribing too long rotations based on historical costs higher than present costs.

It is customary in many enterprises to revalue inventories, and by implication the resources that originally produced them, according to current shifts in reproduction costs, and/or demand. This seems to the writer the most economical procedure, and especially important to follow in forestry with its long rotations.

In the event of a general proportional inflation, the problem is very simple. One converts all data to current prices, all proportions remain fixed, and the optimum rotation remains unmoved.

The interesting problem arises when real or disproportionate price shifts occur. Suppose that stumpage prices remain fixed while regeneration costs increase. The writer submits that one should revalue the historical costs in step with current regeneration costs and pursue the rotation so prescribed.

This procedure may be justified in terms of our first derivation of the Faustmann formula, which proceeded from equating g' with $g\rho + a$. The soil rent, a, conceived as part of the incremental cost of time, is clearly the soil rent of the next rotation. For that is the gain foregone by using land in the present rotation. One should, therefore, compute a in this equation from the regeneration costs, C_o, anticipated at maturity (and from interest rates anticipated over the next rotation).

This done, the C_o in Faustmann's solution (equation 6a) is not the sunk historical cost, but the anticipated cost -- a fact which the writer has sought throughout this paper to intimate by describing it as "regeneration" cost. In working with equation (6a) one may treat regeneration costs as entirely parallel to harvest costs, with which they are virtually simultaneous. Stumpage, g, is already defined net of harvest costs. It is most convenient to redefine it also net of the regeneration costs that must follow immediately on harvest and compress equation (6a) down simply to

$$\rho/\phi = g'/g. \qquad (6a')$$

Critics of the Faustmann formula, as well as some of its expositors, have also alleged that the formula can apply only to new forests commenced from bare land, and not to "going concerns" whose irrevocable sunk costs are obscure and/or irrelevant to current problems. But if the Faustmann formula derives from current or forecasted costs, this criticism is without substance. One may take up the Faustmann reasoning at any point of time.

CHAPTER V

SOME USEFUL INFERENCES FROM THE FAUSTMANN FORMULA

In this section, some of the more significant inferences one may easily draw from the foregoing analyses are outlined briefly. No attempt is made to prove these rigorously, but they will be presented didactically, relying in the main on proofs implicit in preceding sections.

1. Effect of Regeneration Costs on Rotations

Regeneration costs, C_o, tend to lengthen rotations: the higher the costs, the longer the rotations. Harvest costs do likewise, in exactly the same measure, as also do severance taxes.

2. Effect of Interest Rates on Rotations

Contrary to Boulding's theory, interest rates do affect optimum rotations. Higher rates produce shorter rotations. But they do not affect them as much as Allen's theory requires, due to the damping influence of the correction coefficient, Φ.

3. How Much Investment in Regeneration Is Economical?

For every set of regeneration expenditures, there is a yield function. More expenditures are requited generally by both more and earlier yields per acre. Our foregoing analysis implicitly answers the question how much regeneration expenditure is economical. The optimum set of expenditures is that yielding the maximum soil rent (on a Faustmann rotation).

Primary experimental data on the productivity of regeneration expenditures in terms of resultant increased and accelerated yields are woefully deficient. Wider acceptance of one theoretical standard of evaluation would narrow the range of necessary experiments and perhaps make feasible what is now apparently beyond the finances of forestry experiment stations.

4. How Do Taxes Affect Rotations?

The effect of taxes depends very much on the mode of levy. A general property tax levied on standing timber, being a function of timber values, tends obviously to hasten harvests. It also must discourage regeneration and encourage conversion of land to less capital-intensive uses. But once a new stand is started, this tax will again hasten harvest. The forester seeks to minimize the tax burden by working over the years with as little forest capital as he can, which means on short rotations.

Constant annual taxes on site capacity, on the other hand, do not affect optimum rotations in the least. Like other constant annual costs, their effect on the incremental cost of time is just offset by an equal and opposite effect on site rent.

Severance taxes, like other outlays contingent on harvest or regeneration, tend to lengthen rotations.

The effects of income taxes are a little capricious, depending on the individual taxpayer's tax needs. Income taxes may hasten or postpone harvest, depending on when the individual can absorb the income with least tax liability, or use a loss offset to best advantage.

But this capricious effect would not much influence those in high tax brackets. For timber receives "capital gains" treatment: 50 per cent of any gain is non-taxable, and the maximum tax rate on any gain is 25 per cent. This doubtless tends to lengthen rotations, for the capital gains privilege magnifies several times the low percentage gains of postmature timber in eyes that look through the powerful glass of an 80 per cent tax bracket.

5. What Is a Forest's Tax-Paying Capacity?

A forest can pay constant annual taxes up to the amount of its maximum site rent, a. Poorly administered forests could not pay as much as others, but the tax prompts sales from poor administrators to others who impute higher annual site rents. It prompts holders of understocked and overripe stands, yielding little annual growth, to harvest them and commence new stands on a more economical basis.

In general, by making site rent explicit, a constant annual tax encourages timber management such as to maximize site rent. 1/ Of course, taxes higher than site rents imputable by the most efficient users of land would eventually prove uncollectable and the lands revert for taxes.

6. How Much Alternative Income May Be Foregone for Forestry?

1/ Cf. Shirley, p.270. A segment of professional opinion favors levying taxes on this basis. A recent advocate is Bronson (1954). In some jurisdictions, assessors do not revalue trees as they grow, so that the general property tax approximates a constant annual tax on the productive potentiality of the site. Finnish forests are taxed on this basis by design.

Site rent, the annual equivalent of the forest's net yield, is a fitting basis for comparing forestry with other land uses. This is an important by-product of the Faustmann formula. Site rent, a, is directly commensurable with the incremental productivity of land in uses yielding steady annual net incomes. Forestry, to vindicate its tenure, must promise site rents higher than the best alternative so expressed.

Rigorous application of this rule would doubtless reveal that much more or less wooded land should be cleared for other uses (assuming prevailing price levels and without entering the farm "surplus" argument). Many landlords in our economy are, for one reason or another, under only mild constraints to economize on their lands; and lands neglected long enough revert naturally to forest. Probably even some very actively administered forests should be cleared for agriculture.

This may appear as an attack on forestry and "conservation," but it is not intended as such. Conversion of level lands from forest to farm must tend to reduce the economic pressure to clear the hillsides for erosive tilling and increase the economic pressure to reforest them.

7. Are Prevailing Rotations Too Long?

The annual growth of commercial timber in the United States, net of certain losses, expressed as a percentage of the live sawtimber and measured in board feet, in 1952 was very roughly as follows: hardwoods, 4.74 per cent; softwoods, 1.70 per cent; both together, 2.30 per cent. 1/

This growth is the annual yield that must cover not only interest on money tied up in growing stock, but also interest on the site, as well as annual operating charges and taxes. Judged on this basis, an enormous national investment in timber and timberlands hardly seems to be paying its keep. And remember, this is not the marginal growth rate of mature stands alone but the average of all stands of all ages.

Of course the forest produces other values to help justify this otherwise not very productive investment. We lack the data, and

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1/ Statistical Abstract of the United States, 1957, p. 693 cited from U. S. Forest Service Timber Resource Review, preliminary. In terms of cubic feet, the figures become 4.44 per cent, 1.97 per cent, and 2.75 per cent, respectively. Softwoods are generally more valuable, so the aggregate figure should actually be nearer the softwood figure than it is, hence lower.

even the conceptual equipment, to balance these other values in the same scale with timber yields. But it is doubtful if they would bear enough weight to vindicate the present low growth rates because younger forests supply many of the non-timber values more abundantly than old ones. Older forests become "biological deserts," strangling undergrowth and repelling wildlife. 1/ Younger stands offer more both to the sportsman and the cattleman.

In addition to the data cited, there are many indirect reasons for believing rotations are too long. First is the dominance among foresters and economists of doctrines prescribing rotations longer than the optimum. The U. S. Forest Service is the prime example, with its adherence to the maximization of mean annual growth, a zero-interest doctrine. Few foresters seem to commit themselves firmly between the Faustmann Bodenrente solution and the zero-interest Waldrente solution, with its extremely long rotation. Compromise between the two seems to be the prevailing spirit of the forestry literature.

Among economists Fisher's and Allen's solution is popular, as it is with Duerr, Guttenberg, Fedkiw, and other foresters. Jay Gruenfeld, Assistant Land Supervisor with Weyerhaeuser Timber Company, writes: "We do not use $5.00 per acre or any value in determining our rotation period. Particularly due to our heavy volumes of old growth the value per acre does not enter into the determination." 2/

Boulding's solution, it is true, prescribes rotations shorter than the optimum, as does Hildreth's. But these seem to have almost no avowed practitioners.

A second reason is that optimal Faustmann rotations computed from standard yield data, as in Table 2, prove so much shorter than prevailing rotations on comparable sites. This is not conclusive because the "standard" yield data may be based on more comprehensive mensuration and fuller stocking than are economical today. But a third reason is the probable tendency for forest mensuration, which is not after all a costless process, nor one without its obsolete traditions, to lag behind increasing pressures to economize on scarce timberlands. More comprehensive and intensive mensuration leads, as we have seen, to shorter rotations.

And a fourth reason is the prevalance of understocked stands. Due to the "trend toward normality" these often call for longer rotations when, as seems usually to be the case, the site rent, if computed at all, is based on the present rotation rather than the best future rotation. Economy of the site would call for borrowing the site rent value from the best future rotation, which would tend to clip rotations on understocked stands very short.

Fifth is the widespread practice of holding timberlands for motives ulterior to timber culture: for mineral rights, water

1/ Dolder, 1955, pp. 50 ff.
2/ Letter of March 20, 1957, author's files.

rights, price increments to the land, political control, and so on. Holders with these ulterior motives, and like as not with outside funds, are under little pressure to make the most of the timber-growing capacity of their lands. Often they accept whatever regeneration Nature offers spontaneously, which tends to be too little and too late for an optimal rotation.

Landholders thus freed from economic constraints to economize on timber sites might conceivably depart from optimal rotations on the short as well as the long side. But their ample low-interest funds make it unlikely they would care about maximizing their internal rates of return. They generally would err on the long side.

A sixth reason is the probability of physical damage to aged timber, a probability not reflected in standard yield tables. The forester might well paraphrase Biblical advice: "Lay not up treasures in tall timber, which moth and rust do corrupt and fires break through and destroy." Adding an annual charge for the probability of such loss, increasing with age, would tend to shorten rotations even below those derived from standard yield tables.

A seventh is the Federal income tax law, which exempts timber from ordinary tax rates by allowing it "capital gains" treatment. This makes the low percentage yields of aged timber look quite agreeable to those in high tax brackets. To the individual of course this does justify the longer rotation. We are here considering what is best for the whole society.

On the other hand we should entertain three arguments suggesting that rotations are not, after all, too long.

The first is that our local general property taxes, falling in part on standing timber, create an artificial incentive to shorten rotations below the optimum. There is good reasoning in this proposition. But remember that timber does not necessarily escape taxation when harvested. Log decks are taxed too, and, more important, buildings and many other durable wood products. In many jurisdictions the general property tax on standing timber is purely nominal, thanks to obsolete assessments. In others it has been replaced by statute by severance taxes, which tend to postpone harvest. But taxes on buildings have hardly anywhere been lifted. Public buildings, it is true, are tax exempt, but they are not noted for heavy use of wood, whereas tax-exempt National Forests are indeed noted for their heavy stands of timber. On balance, it would not be easy to generalize that property taxation, considered in all its aspects, tends unduly to shorten rotations.

Second is the argument that timber is held for the price increment, as though price increment were something as confidently to be expected as physical growth. But this assumption is unwarranted.

In the first place, the mere depreciation of the dollar is no argument for holding timber in preference to other equities. The only price increment to consider seriously is relative price increment, that is price increment deflated by a general price index.

Over 80 years in England, Hiley reports that the corrected price relative for imported sawn wood rose 0.5 per cent per annum, from 1857 to 1937 (Hiley, 1955). That is too little to justify any great lengthening of rotation ages over those prescribed by physical growth at constant price levels.

Do 80 years of past English experience tell us what to expect here in the next 80? No one can read the future, but a knowledge of the past helps. While there are times and places of spectacular advances of relative timber prices, there are also stunning setbacks, such as we are witnessing today. There is increasing pressure on limited timberlands, true, but there is also increasing pressure on other limited natural resources, and there is a dynamic, innovating economy to reckon with, continually generating new substitutes for overpriced resources. Timber has proved to be very substitute-prone, making a mockery of the predictions circa 1907 of acute timber famine, after which prices collapsed abysmally.

In the perspective of history, overestimation of future price increments to standing timber is not only possible, but quite characteristic of the boom phase of a business cycle. Booms increase demands on limited natural resources and generate Malthusian anxieties that lead to timber-hoarding on an unreasonable scale.

Around the turn of the century Gifford Pinchot's dire warnings of imminent timber bankruptcy seem to have helped convince many persons that timber speculation was the royal road to riches. The subsequent famous 1911 Bureau of Corporations report on The Lumber Industry, persuasively foreboding early monopolization of remaining virgin stands, must also have deterred many hopeful investors from releasing supplies.

Timber speculation such as that would have been quite rational, and to a degree socially useful, were a future shortage truly imminent. But in an imperfect, ill-advised market, speculation can and did proceed with precious little cognizance of supply-demand relations in the long run.

Speculative timber withdrawals hold a price umbrella over the market and thus encourage development of new supplies 1/ by means of new access roads, more intensive land management, increased imports, milling advances, and so on. For all of these advances our loose and wasteful land economy offers great scope. Briegleb, for example, stated in 1956 that "22 million acres of idle land in the South should be planted to trees," in an article entitled "South's Timber Crop Could be Doubled" (Briegleb, 1956). Let these new supplies hit the market simultaneously with the speculative holdings, let construction recess about the same time, and the optimistic timber holder can only wish he had gathered his

1/ Cf. Marquis, pp. 38-39.

rosebuds while he might.

By hindsight from 1957 it seems clear that the Malthusian climate of opinion of the last few years may already have led to excess supplies similar to those of 1907 and 1929. Now the argument becomes "The present market is weak, so the rational course is to wait for the upturn which increasing population pressures will inevitably bring."

But unless one anticipates that resurgent demand quite quickly, that is an argument for harvesting. If it is the "inevitable, ultimate" Malthusian triumph that one awaits, some decades away, he had better put a higher value on the site. This would prompt him to harvest the present stand and begin the next.

As distinguished a forester as Hiley made the slip of including increments to the price of forest land with increments to the price of timber in computing the financial yield of a timber rotation (Hiley, 1955, p.6). If one does this, then shortages anticipated in the remote future, whose main present manifestation is to increase the price of timber sites, would indeed justify longer rotations. But it must be quite evident that site price increments occur whether timber is harvested or not, and are not sold with the harvested timber, so it would be folly to delay harvest because site prices are rising. On the contrary: this argues for higher carrying costs on the site which lead to shorter rotations.

A third argument that rotations are not too long is that shorter rotations would mean a flood of increased output and spoil the market. The monopolistic tone of this argument is evident, and from the social viewpoint it has therefore no merit. Even from the monopolist's viewpoint it defeats itself in the long run, since longer rotations up to a point (W' on Fig. 1) mean higher mean annual yields.

In summary, then, there are several reasons to believe that timber rotations tend to be too long. But this is not to say they all are, nor to deny that some are uneconomically short. Indeed, the most glaring diseconomy in forestry is the extreme contrast between credit-starved small holders who harvest young trees while they are still growing very rapidly, and large public and corporate holders who keep stagnant decaying virgin timber off the market. The contrast offers some measure of the failure of our forest economy to achieve optimal allocation of its resources.

CHAPTER VI

CONTRIBUTIONS OF THIS STUDY

The main theme of this study is that Faustmann's concept of financial maturity has greater merit than the others analyzed. While it is useful to rally to support an embattled truth, this is no new intellectual contribution. Much other material is also borrowed. What, then, can the study show as contributions to forestry and economics?

1. It shows that marginal analysis is valid and useful for handling time relationships in economics and refutes Boulding's allegation to the contrary. Besides, it develops in some depth a marginal analysis of one economic problem involving time, in both its analytical and applied aspects, and thereby suggests one way economists might apply marginal analysis to economic problems involving time, demonstrating that marginal techniques are not only practicable, but more so than alternative techniques.

2. The study shows the consistency of marginal analysis with Faustmann's concept of financial maturity.

3. It brings Faustmann's formula to the attention of economists, not just as a neglected historical curiosity but as an important useful tool. It shows the formula's necessity in terms not only of marginal analysis, but several other viewpoints likely to commend themselves to economists with different methodological preferences.

Thus the study removes any question that a valid concept of financial maturity may vary with individual caprice, and translates several variant economic dialects into one another. Also it enhances the operator's understanding and hence his ability to apply the formula intelligently under a wide range of conditions and avoid "formula-feeding" which inevitably leads to abuses and errors.

4. It brings the possibilities of this aspect of marginal analysis to the attention of foresters, confirming Faustmann's formula and offering several working advantages over the maximization technique for applying it: marginal analysis permits a sharper definition of revenues and costs at the critical time of the harvest decision; by annualizing site rent and treating it as part of the marginal cost of time, it frees Faustmann's formula from dependence on a cumbersome assumption of an infinite future chain of forest crops; most important of all, it allows easier flexibility in adapting the formula where data change in mid-rotation and where stands are understocked (see point 5, below).

5. The study shows the technique of borrowing the site rent value from the future rotation, where that differs from the present, that is, in cases of data changing in mid-rotation, and understocked stands. It refutes the criticism that Faustmann's formula is necessarily based on irrelevant historical costs -- and to caution against

rigid blind formula-feeding that would warrant the criticism.

This is probably the most important advantage of marginal analysis over the traditional forestry concept of Faustmann's formula as a maximization of soil expectation value. Marginal analysis is more adaptable to dynamic conditions, and that is the difference between a stiff mathematical exercise and a useful working tool.

6. It shows the necessity for treating the site separately from other inputs in problems of time economics, due to the site's not being embodied in the product. This is necessary whether one maximizes the net return to the site or to some other input.

7. It shows that it is generally advantageous to treat the site as the residual claimant on the product, not for any traditional or mystical reason but because it is harder to put an external value on the site than on other inputs. It is valid to maximize residual net yields to other inputs where these conditions are reversed.

8. The study shows the necessity for accurate imputation of the product among specific tangible inputs in order to achieve rational allocation and management of resources through time. It shows that reliance on a vaguely defined residual catch-all, "profits," leads to equivocal imputation and probable errors of management. It demonstrates how one can impute "profits" to specific inputs under a wide range of conditions, and can integrate functional distribution theory in a specific time problem with production theory as an example of how the two can work together.

9. The study refutes Allen's and Fisher's and Boulding's concepts of financial maturity.

10. The study also refutes recurrent efforts to show that Faustmann's solution is the same as some other in general (Boulding), in the case of overlapping rotations (Lutz), or of limited time-horizons (Lutz), or in perfect competition (Lutz, Scitovsky).

11. It salvages something of Boulding's solution by showing it is legitimate to maximize i provided it is net of soil rent. It points to the pitfall of a dual rate of return, shows the means of avoiding it, and thus arrives at a new concept of financial maturity, joint maximization of i and a, that is superior to Faustmann's in certain limited conditions.

12. The study makes of Faustmann's formula a complete general solution to the economic problems of replacement and turnover. This is achieved first, by using marginal analysis to adapt it to mid-rotation changes of data, as already mentioned; second, by applying it to depreciating assets with little or no salvage value such as fruit trees and buildings.

It also points out the inadequacy of current concepts of financial maturity of depreciating assets, even where site rent is included with the marginal cost of time, because of the need to find site rent and financial maturity simultaneously.

13. It points to the need for a valid concept of financial maturity in order to judge among different land uses. For to judge we must reduce various land use plans, yielding future incomes differently distributed in time, to a commensurable annual equivalent; and to do that, we must find financial maturity.

14. It works out an easily operable technique for finding Faustmann's solution in practice, through the use of tables of ρ/ϕ, demonstrates use of the technique and presents many actual solutions worked out with it. It also shows that one may simplify the operation by omitting constant annual costs and revenues.

15. The study refutes the argument of Duerr and others that Faustmann's solution is not significantly different from Allen's and Fisher's in practice. It also outlines the conditions when they differ significantly, analyzing the influence of site, interest rate, harvest and regeneration cost, species, mensuration standards, stocking, and price anticipations.

16. It calls attention to the indefensible concept of financial maturity used by the U.S. Forest Service in evaluating forest practices on private lands, and the concept's bias against smaller forest holders.

17. It uses Faustmann's formula in analyzing several questions with a practical bearing on private and public economic policy:

(a). Working out the market value of immature timber separately from its site;
(b). Showing that higher regeneration costs prescribe longer rotations, contrary to a common impression;
(c). Showing that higher interest rates shorten optimum rotations, contrary to Boulding's analysis, but that they do not shorten them as much as implied in Allen's and Fisher's analysis;
(d). Pointing out that the optimum regeneration expenditure is simply that which maximizes a -- a proposition which seems self-evident until one surveys some of the literature that makes something artificially complex out of this problem (e.g. Streyffert);
(e). Pointing out that severance taxes lengthen rotations; property taxes on standing timber shorten them, <u>ceteris paribus</u>; constant annual taxes on the site leave optimal rotations unchanged, but apply an external lever prompting unenterprising landholders to adopt plans maximizing soil rent; and that a forest's maximum long run tax paying capacity is the annual site rent imputed by the most capable land manager.
(f) Pointing out that rotations are probably too long. This is important not just as a technical observation but as an index to the inadequacy of the institutional framework on which we rely to bring economic pressures to bear on landholders to put the resource to its most productive uses.

18. The study supplies ammunition to inter-disciplinarians by pointing up the noncommunication between forestry and economics that has let Faustmann's formula, a commonplace among foresters for a century, go unnoticed by economists, including the Austrians who read Faustmann's language, lived nearby, and were obsessed for decades with the question of the "period of production," as exemplified by timber culture.

19. It points up areas in which future research might be productive. This we treat separately in Chapter VII.

CHAPTER VII

SUGGESTED FUTURE RESEARCH

In the course of this study several fruitful lines for future research have appeared. Some of these are:

1. In forestry economics

(a). To re-evaluate the U.S. Forest Service judgments of the merit of various classes of timber landholders in terms of Faustmann's concept of financial maturity.

(b). To reconsider rotation policies on public lands in the same terms.

(c). To integrate forest economics with mill-town, mill, and transportation economics to work out rotations that are optimal in terms of all the economic forces considered together.

(d). To investigate to what extent economies to scale in large mills are achieved at the expense of diseconomies in company forests.

(e). To work out monetary yield tables tailored to local markets.

(f). To put monetary values on non-timber forest benefits and integrate these with timber yields in working out optimal rotations.

(g). To work out probabilities of physical damage, including catastrophic loss, for use in determining optimal rotations.

(h). To work out an optimal system of forest taxation.

(i). To explain why many actual rotations are so much longer than optimal rotations worked out from standard yield data.

(j). To explain why market values for forest sites are often lower than soil expectation values based on standard yield tables and market interest rates. The explanation that business firms require returns much higher than market rates calls for critical examination, inasmuch as companies requiring very high returns would not hold timber at all.

(k). To develop figures on regeneration and harvest costs corresponding to given yield tables, and recompute financial maturity on this basis;

(l). To investigate the effect of understocking on rotations; to work out a new concept of "normal" or "standard" yields based on an economic standard of stocking.

2. In general

(a). To apply Faustmann's formula to specific problems of replacement and turnover;

(b). To analyze the macro-economic implications of applying Faustmann's formula to speed the turnover of the economy's total capital stock, which would tend to increase income and employment.

CHAPTER VIII

CONCLUSION

Faustmann's formula represents an important advance over the concepts of financial maturity advanced by several economists, and by those foresters who would dispense with interest costs. It is time both professions acknowledged its fundamental contribution.

But Faustmann's formula is no final solution to the question of financial maturity. It is only a simple classic theme, if you will, for latter day composers to weave into larger patterns, to vary and orchestrate with the full symphony of modern instruments.

This paper has purported to enhance its usefulness, but has said nothing of programming, of probability and expectation, and many other kinds of analyses that might be joined with Faustmann's formula. But of all the things this study has had to neglect, the most important by far are the macro-economic implications. Faustmann's concept of financial maturity prescribes policies of rapid turnover and replacement -- more rapid, if we may generalize from our forestry studies, than are customarily practiced today. General application of Faustmann's formula in all industries would, if this is so, speed the turnover of the nation's capital stock, which in turn would contribute toward increasing employment and income. 1/ Should economists find Faustmann's formula acceptable and devise public policies to prompt its practical application, this benefit would, from the stand point of human welfare, probably outweigh all the other benefits.

1/ Cf. Wicksell, pp. 172 ff.

APPENDIX A

ALTERNATIVE DERIVATIONS OF FAUSTMANN'S FORMULA

The first textual derivation sacrifices something of mathematical perfection for simplicity and clarity of exposition. Hoping it has served its function, we will improve on it a bit for more exacting readers. One may protest that investment in timber, interest on which we reckon as part of the incremental cost of time, is generally greater than the stumpage value (g). After the first several years, the stumpage value may still be absolutely nothing even though one has invested in his trees the regeneration costs, several years' interest on them, and several years' use of the forest site, and could in a perfect market recover his investment by selling his saplings, not for immediate harvest, but for the buyer to hold to maturity. 1/ There is, indeed, only one point of time when g rises as high as the accumulated investment in the timber. That time is the time of financial maturity. Before and after this, it is less. This follows from our putting on the annual use of land a value, a, so high that one can realize it only by harvesting at the optimum time.

The first textual derivation yields the correct result because at that optimum harvest time g does equal the investment in the timber, so that $g\rho + a$ is the full incremental cost. But it is somewhat disquieting for a major conclusion to depend on this coincidence, and to note such anomalies as that the sum of the incremental costs, as defined, falls far short of the total costs.

Let us then figure the incremental cost of time again, replacing the stumpage (g) with the accumulated investment in the tree. This latter is the total cost at any time.

$$\text{Total cost} = C_o(1+i)^t + a\,\frac{(1+i)^t - 1}{\rho} \qquad (7)$$

Incremental cost now equals interest on the total investment plus the annual value of the site (a).

$$\text{Incremental cost} = a + \rho\left[C_o(1+i)^t + a\,\frac{(1+i)^t - 1}{\rho}\right] \qquad (8)$$

1/ While markets for standing timber are far from perfect, still "the day is fast passing when young stands can be purchased for 'little or nothing'." Roy B. Thompson, An Examination of Basic Principles of Comparative Forest Valuation (Duke University School of Forestry, Bulletin No. 6, 1942), 73. The Federal Reserve Bank of Richmond noted in its Monthly Review for December, 1956, page 6, that "...planted trees always add to the sale value of the land."

This reduces to,

$$\text{Incremental cost} = C_o\ \rho\ (1+i)^t + a\ (1+i)^t$$

$$= (1+i)^t\ (C_o\rho + a) \qquad (8a)$$

Equating this incremental cost with the incremental product of time (g'), and solving for t, we get, as before,

$$t = \frac{1}{\rho} \ln \frac{g'}{g' - \rho\ (g-C_o)} \qquad (6)$$

Another way to conceive and formulate equation (8a) is to differentiate total cost to find incremental cost. Total cost, as previously defined in equation (7), is the compounded value of regeneration costs, plus the accumulated and compounded value of the use of the site. Differentiating, we obtain again,

$$\text{Incremental cost} = C_o\ \rho(1+i)^t + a(1+i)^t$$

$$= (1+i)^t\ (C_o\rho + a) \qquad (8a)$$

Whence we may proceed as above to the Faustmann solution of equation (6).

Boulding, as we mentioned, has disavowed the marginalist approach in solving this problem. In view of this, and also of the more general anti-marginalist sentiment that flares up from time to time, it may be well to confirm the foregoing derivations with others not proceeding from an initial equation of incremental rates.

Another approach, one that Boulding himself uses, is simply to maximize a. Boulding comes to grief, as we saw, by trying to do it with a numerical example. It is simpler and surer to maximize a by setting its derivative equal to zero.

$$\frac{da}{dt} = i\ \frac{(1+i)^t\ [g' - (g-C_o)] - g'}{[(1+i)^t - 1]^2} = 0 \qquad (3')$$

Solving for t, we obtain once again the Faustmann solution:

$$t = \frac{1}{\rho} \ln \frac{g'}{g' - \rho\ (g - C_o)} \qquad (6)$$

The German forester, Martin Faustmann, who first published his formula in 1849, 1/ derived it by yet another method, one which does not involve the annualization formula (equations 1, 1a, and 3). Faustmann maximized what came to be called Bodenerwartungswerte, or "soil expectation value," the selling price of land derived from summing the infinite series of discounted net yields expected from future rotations. "Soil expectation value" in English language forestry texts is generally designated S_e.

$$S_e = \frac{g - C_o(1+i)^t}{(1+i)^t} + \frac{g - C_o(1+i)^t}{(1+i)^{2t}} + \ldots + \frac{g - C_o(1+i)^t}{(1+i)^{nt}} + \ldots$$

$$= \frac{g - C_o(1+i)^t}{(1+i)^t - 1} \quad 2/ \qquad (9)$$

This "soil expectation value," note, is the same as the "soil rent," the a of our calculations, divided by the continuous interest rate, ρ.

$$S_e = a/\rho \qquad (10)$$

Soil expectation value, that is, is in effect soil rent capitalized in the familiar way:

$$S_e \cong a/i \qquad (10a)$$

ρ not being a function of time, S_e and a reach their maxima simultaneously.

Two other derivations follow from stating the problem this way: current tree growth should earn a return on the value of the tree plus the value of the site. The first is quite direct:

1/ Martin Faustmann, "Berechnung des Werthes, welchen Waldboden, sowie noch nicht haubare Holzbestände für die Waldwirthschaft besitzen," Allg. Forst und Jagd Zeitung, 25 (1849), 441-455) The formula first appears on p. 442, where Faustmann actually derives it by two methods, one of them using the annualization formula. However, it is the other method that has come to be associated with his name.

2/ Most standard forestry texts carry the Faustmann formula in this form. W. E. Hiley, Woodland Management (London: Faber and Faber, Ltd., 1954), 124; W. Schlich, Forest Management, Vol. 3 of Schlich's Manual of Forestry (2d ed. rev.; London: Bradbury, Agnew and Co., Ltd., 1905), 124. Our version here is simpler than that usually presented because intermediate costs and revenues are assumed away.

$$g' = \rho\,(g + V) = \rho\left[\frac{g\,[(1+i)^t - 1] + g - C_o(1+i)^t}{(1+i)^t - 1}\right] = \rho\,\frac{g - C_o}{\phi} \tag{11}$$

But this is equation (6) again, in the operable form (6a) devised by Duerr et al. (pp. 15,27).

The second lumps V with C_o as part of the original input at t_o on which the forest earns a return. Then, as the site is not sold with the product, we must count the unexhausted value of the site at harvest time as part of the product or deduct it from costs. Then we state the yield, g, as equal to the sum of the costs, and maximize V, which also maximizes $a = Vi$ (so long as i is fixed):

$$g = (C_o + V)(1+i)^t - V; \qquad V = \frac{g - C_o(1+i)^t}{(1+i)^t - 1} \tag{12}$$

Setting the time-derivative of V equal to zero and solving for t, we get once again the Faustmann solution. We need not repeat the operation, which is virtually the same as for maximizing a, equation (9) above.

APPENDIX B

LIST OF EQUATIONS

	Equation	Pages Discussed
(1)	$a = \dfrac{Ai}{(1+i)^t - 1}$	p. 5
(1a)	$a = \dfrac{A\rho}{(1+i)^t - 1}$	p. 6
(2)	$A = g - C_o(1+i)^t$	p. 6
(3)	$a = \rho \left[\dfrac{g - C_o(1+i)^t}{(1+i)^t - 1}\right]$	p. 6
(3')	$\dfrac{da}{dt} = i \dfrac{(1+i)^t \; [g' - \rho(g-C_o)] - g'}{[(1+i)^t - 1]^2}$	p. 91
(3a)	$a = \left[\dfrac{g - C_o(1+i)^t}{(1+i)^t - 1}\right] \rho + R - c$	p. 69
(4)	$dg/dt = g\rho + a$	pp.7,19,20
(5)	$g' = g\rho + \left[\dfrac{g - C_o(1+i)^t}{(1+i)^t - 1}\right] \rho$	p. 7
(5a)	$g' = \rho \dfrac{(1+i)^t \; (g-C_o)}{(1+i)^t - 1}$	p. 9
(6)	$t = \dfrac{1}{\rho} \ln \dfrac{g'}{g' - \rho \; (g-C_o)}$	pp.7,9,47,91
(6a)	$\rho = \dfrac{g'}{g-C_o} \; \dfrac{(1+i)^t - 1}{(1+i)^t}$	pp. 9, 25
(6a')	$\dfrac{\rho}{\phi} = \dfrac{g'}{g}$	p. 76
(6b)	See equation (11) p. 93	
(6c)	$\dfrac{\rho}{\phi} = \dfrac{g'_m + R_m}{g_m + \sum_{o}^{m} R_t \; (1+i)^{-t}}$	p. 70
(6d)	$\dfrac{\rho}{\phi} = \dfrac{R_m}{\sum^{m} R_t \; (1+i)^{-t}}$	p. 74

(7) $\text{Total cost} = C_o(1+i)^t + a\left[\frac{(1+i)^t - 1}{\rho}\right]$ pp. 7, 90

(7a) $T_o = \frac{g - a\frac{(1+i)^t - 1}{\rho}}{(1+i)^t}$ pp. 7,19

(8) $\text{Incremental cost} = a + \rho\left[C_o(1+i)^t + a\left[\frac{(1+i)^t - 1}{\rho}\right]\right]$ p. 90

(8a) $\text{Incremental cost} = C_o\rho\ (1+i)^t + a(1+i)^t = (1+i)^t\ (C_o\rho + a)$ p. 91

(9) $S_e = \frac{g - C_o(1+i)^t}{(1+i)^t} + \frac{g - C_o(1+i)^t}{(1+i)^{2t}} + \ldots$

$+ \frac{g - C_o(1+i)^t}{(1+i)^{nt}} + \ldots + \frac{g - C_o(1+i)^t}{(1+i)^{\infty}}$

$= \frac{g - C_o(1+i)^t}{(1+i)^t - 1}$ p. 92

(10) $S_e = a/\rho$

(10a) $S_e \cong a/i$

(11) $g' = \rho(g+V) = \rho\left[\frac{g\ [(1+i)^t - 1] + g - C_o(1+i)^t}{(1+i)^t - 1}\right]$

$= \rho\ \frac{g - C_o}{\phi}$ p. 93

(12) $g = (C_o + V)\ (1+i)^t - V;\ V = \frac{g - C_o(1+i)^t}{(1+i)^t - 1}$ pp. 53, 93

(13) $\phi(t) = \frac{(1+i)^t - 1}{(1+i)^t} = 1 - \frac{1}{(1+i)^t}$ p. 10

(14) $g'/g = \rho$ pp. 17, 25

(14a) $g' = g\rho$ p. 19

(15) $\frac{g - C_o(1+i)^t}{(1+i)^t} = \frac{g}{(1+i)^t} - C_o$ p. 18

(16) $i_B = \sqrt[t]{\frac{g}{C_o}} - 1$ p. 45

(17) $t_B = \frac{g}{g'} \ln \frac{g}{C_o}$ p. 48

(18) $\frac{T'}{T} = \frac{C_o\rho + a_F}{C_o + \frac{a_F}{\rho}\left[1 - \frac{1}{(1+i)^t}\right]}$ p. 49

(19) $J = \frac{g - C_o(1+i)^t}{t}$ p. 50

(20) $t_J = \frac{1}{\rho} \ln \frac{g - tg'}{C_o(1-t\rho)}$ p. 50

(21) $t = \frac{g + V}{g'} \ln \frac{g + V}{C_o + V}$ p. 54

(22) $a_H = \frac{g}{t(1+i)^t}$ p. 59

(23) $g' = g\rho + \frac{g}{t}$ p. 59

(24) $V = \frac{g-C_e}{t}$ p. 61

Values of $\frac{\rho}{\phi}$, Where $\rho = \ln(1+i)$ and $\phi = \frac{(1+i)^t - 1}{(1+i)^t}$

i ρ	0.25% .00250	0.50% .00499	0.75% .00748	1.00% .00995	2.00% .01980	3.00% .02956	4.00% .03922	5.00% .04879	6.00% .05827	7.00% .06766	8.00% .07696
t= 1	100.2	100.3	100.5	100.5	101.0	101.5	102.0	102.5	102.9	103.4	103.9
2	50.2	50.3	50.4	50.5	51.0	51.5	52.0	52.5	53.0	53.5	53.9
3	33.5	33.6	33.7	33.8	34.3	34.8	35.3	35.8	36.3	36.8	37.3
4	25.2	25.3	25.4	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0
5	20.2	20.3	20.4	20.5	21.0	21.5	22.0	22.5	23.1	23.6	24.1
6	16.8	16.9	17.1	17.2	17.7	18.2	18.7	19.2	19.7	20.3	20.8
7	14.4	14.5	14.7	14.8	15.3	15.8	16.3	16.9	17.4	17.9	18.5
8	12.6	12.8	12.9	13.0	13.5	14.0	14.6	15.1	15.6	16.2	16.7
9	11.3	11.4	11.5	11.6	12.1	12.7	13.2	13.7	14.3	14.8	15.4
10	10.1	10.3	10.4	10.5	11.0	11.6	12.1	12.6	13.2	13.8	14.3
11	9.23	9.35	9.48	9.60	10.12	10.65	11.19	11.75	12.31	12.89	13.48
12	8.47	8.59	8.72	8.84	9.36	9.80	10.45	11.01	11.58	12.17	12.77
13	7.83	7.95	8.08	8.20	8.72	9.27	9.82	10.39	10.97	11.57	12.17
14	7.28	7.40	7.53	7.65	8.18	8.72	9.28	9.86	10.45	11.05	11.67
15	6.80	6.92	7.05	7.18	7.70	8.25	8.82	9.40	10.00	10.61	11.24
16	6.38	6.48	6.62	6.77	7.28	7.84	8.42	9.00	9.62	10.24	10.87
17	6.01	6.16	6.29	6.38	6.92	7.48	8.05	8.65	9.26	9.91	10.54
18	5.69	5.80	5.94	6.07	6.60	7.16	7.75	8.35	8.96	9.61	10.26
19	5.40	5.54	5.67	5.78	6.31	6.87	7.47	8.08	8.71	9.36	10.02
20	5.13	5.25	5.38	5.53	6.06	6.63	7.21	7.83	8.47	9.12	9.80
21	4.89	5.04	5.16	5.26	5.82	6.40	6.99	7.61	8.25	8.93	9.61
22	4.68	4.80	4.92	5.05	5.61	6.18	6.79	7.41	8.07	8.74	9.43
23	4.48	4.62	4.73	4.85	5.41	6.00	6.60	7.24	7.90	8.58	9.27
24	4.29	4.42	4.56	4.69	5.24	5.82	6.43	7.07	7.74	8.43	9.14
25	4.13	4.26	4.40	4.52	5.08	5.66	6.28	6.92	7.60	8.29	9.01
26	3.97	4.09	4.23	4.36	4.92	5.51	6.14	6.79	7.47	8.17	8.90
27	3.83	3.96	4.09	4.22	4.78	5.37	6.01	6.66	7.35	8.06	8.80
28	3.70	3.84	3.96	4.09	4.65	5.25	5.88	6.55	7.25	7.96	8.71
29	3.58	3.70	3.84	3.96	4.53	5.13	5.78	6.44	7.15	7.88	8.62
30	3.46	3.59	3.72	3.86	4.42	5.03	5.67	6.34	7.05	7.79	8.54
31	3.35	3.49	3.61	3.75	4.31	4.93	5.57	6.26	6.97	7.71	8.48
32	3.25	3.37	3.51	3.64	4.22	4.83	5.48	6.18	6.90	7.65	8.41

(Continued)

Values of $\frac{\rho}{\phi}$, Where $\rho = \ln(1+i)$ and $\phi = \frac{(1+i)^t - 1}{(1+i)^t}$ (Continued)

i / ρ	0.25% .00250	0.50% .00499	0.75% .00748	1.00% .00995	2.00% .01980	3.00% .02956	4.00% .03922	5.00% .04879	6.00% .05827	7.00% .06766	8.00% .07696
t=33	3.16	3.28	3.42	3.55	4.12	4.74	5.40	6.10	6.82	7.58	8.36
34	3.07	3.20	3.34	3.47	4.04	4.66	5.33	6.02	6.76	7.52	8.30
35	2.99	3.12	3.25	3.38	3.96	4.58	5.25	5.96	6.70	7.47	8.26
36	2.91	3.04	3.17	3.31	3.88	4.51	5.19	5.90	6.64	7.42	8.21
37	2.83	2.95	3.09	3.23	3.82	4.45	5.12	5.84	6.59	7.37	8.17
38	2.76	2.88	3.03	3.16	3.74	4.38	5.06	5.79	6.54	7.32	8.14
39	2.69	2.82	2.96	3.09	3.68	4.32	5.01	5.73	6.50	7.28	8.10
40	2.63	2.76	2.90	3.03	3.62	4.27	4.95	5.69	6.45	7.25	8.07
41	2.57	2.70	2.83	2.97	3.56	4.21	4.90	5.64	6.42	7.21	8.04
42	2.51	2.64	2.78	2.91	3.50	4.16	4.86	5.60	6.38	7.18	8.01
43	2.45	2.59	2.72	2.86	3.45	4.11	4.81	5.56	6.35	7.16	7.99
44	2.40	2.53	2.67	2.80	3.40	4.06	4.77	5.52	6.31	7.13	7.97
45	2.35	2.48	2.62	2.76	3.36	4.02	4.73	5.49	6.29	7.11	7.94
46	2.30	2.43	2.57	2.71	3.31	3.98	4.70	5.46	6.26	7.08	7.93
47	2.26	2.39	2.53	2.66	3.27	3.94	4.66	5.43	6.23	7.06	7.91
48	2.21	2.34	2.49	2.62	3.23	3.90	4.62	5.40	6.21	7.04	7.89
49	2.17	2.30	2.44	2.58	3.19	3.86	4.59	5.37	6.19	7.02	7.88
50	2.13	2.26	2.40	2.54	3.15	3.83	4.57	5.34	6.16	7.00	7.86
51	2.09	2.22	2.36	2.50	3.11	3.79	4.53	5.32	6.14	6.99	7.85
52	2.05	2.19	2.32	2.46	3.08	3.77	4.51	5.30	6.12	6.98	7.84
53	2.02	2.15	2.29	2.43	3.05	3.74	4.48	5.27	6.11	6.96	7.83
54	1.98	2.11	2.25	2.39	3.01	3.71	4.46	5.26	6.09	6.95	7.82
55	1.95	2.08	2.22	2.36	2.99	3.68	4.44	5.23	6.08	6.93	7.81
56	1.91	2.05	2.19	2.33	2.96	3.65	4.41	5.22	6.06	6.93	7.80
57	1.88	2.02	2.16	2.30	2.92	3.63	4.39	5.20	6.04	6.91	7.79
58	1.85	1.99	2.12	2.27	2.90	3.60	4.37	5.18	6.03	6.90	7.79
59	1.82	1.96	2.10	2.24	2.87	3.58	4.35	5.17	6.02	6.89	7.78
60	1.80	1.93	2.07	2.21	2.85	3.56	4.33	5.16	6.01	6.88	7.77
61	1.77	1.90	2.04	2.19	2.82	3.54	4.31	5.14	6.00	6.88	7.77
62	1.74	1.88	2.02	2.16	2.80	3.52	4.30	5.13	5.99	6.87	7.76
63	1.72	1.85	1.99	2.14	2.78	3.50	4.29	5.11	5.98	6.86	7.76
64	1.69	1.83	1.97	2.11	2.76	3.48	4.27	5.10	5.97	6.86	7.75

(Continued)

Values of $\frac{\rho}{\phi}$, Where $\rho = \ln(1+i)$ and $\phi = \frac{(1+i)^t - 1}{(1+i)^t}$ (Continued)

i	0.25%	0.50%	0.75%	1.00%	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	8.00%
ρ	.00250	.00499	.00748	.00995	.01980	.02956	.03922	.04879	.05827	.06766	.07696
t=65	1.67	1.80	1.94	2.09	2.73	3.46	4.25	5.09	5.96	6.85	7.75
66	1.65	1.78	1.92	2.07	2.72	3.45	4.24	5.08	5.95	6.84	7.74
67	1.62	1.76	1.90	2.04	2.69	3.43	4.23	5.07	5.95	6.84	7.74
68	1.60	1.73	1.88	2.02	2.68	3.41	4.21	5.06	5.94	6.83	7.73
69	1.58	1.71	1.86	2.00	2.66	3.40	4.20	5.06	5.93	6.83	7.73
70	1.56	1.69	1.84	1.98	2.64	3.38	4.19	5.05	5.93	6.83	7.73
75	1.46	1.60	1.74	1.89	2.56	3.32	4.14	5.01	5.90	6.81	7.72
80	1.38	1.52	1.66	1.81	2.49	3.26	4.10	4.98	5.88	6.79	7.71

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