Chapter 3. The von Thünen Model and Land Rent Formation

Masahisa Fujita and Jacques-François Thisse

March 1, 2012

1 INTRODUCTION

Land use models explain the way various activities using land locate over a given area. This phenomenon can be studied from a different perspective by asking which activities are accommodated in specific locations. As will be seen in this chapter, these two approaches may be considered interchangeable, although they differ somewhat. The first is more in line with microeconomics in that the analysis focuses on where given agents chose to locate, whereas the second is more akin to the approach followed by many geographers, who put the emphasis on places and densities and not on agents.[1]

Because, in a market economy, land is allocated among activities through the price of land, the land use problem is equivalent to asking how the price of land is determined in a competitive economy. This does not seem to be a feasible task, for we have just seen that the price mechanism does not work in a spatial economy. The spatial impossibility theorem does not preclude, however, the possibility of uncovering particular, but relevant, economic situations in which the price mechanism is able to govern the allocation of activities over space. This is precisely what we will try to do in this chapter.

The prototype of such particular situations has been put forward by von Thünen (1826), who sought to explain the pattern of agricultural activities surrounding cities in preindustrial Germany. His model relies on the following basic idea: *each farmer faces a trade-off between land rents and*

[1] Either approach will be used in this book. Roughly speaking, we can say that the former is followed in models with a finite number of agents to locate, while the latter will be encountered in models with a continuum of agents.

transport costs. The various models developed in his footsteps can be cast within the Arrow–Debreu framework because transactions must occur at a given marketplace (the town in Thünen's analysis), whereas activities (the crops in Thünen's analysis) and land are supposed to be perfectly divisible. Once markets are considered as perfectly competitive, it becomes easy to understand why the Thünian model has been extensively studied in both production theory and urban economics where it has proven to be a very powerful tool. That is, the Thünian model rests on the paradigmatic combination formed by the standard assumptions of constant returns and perfect competition, while assuming an exogenously located marketplace.

Each location in space is a bundle of characteristics such as soil conditions, relief, geographical position, and the like. Both land rent and land use vary across locations depending on these characteristics. Among them, the most important for location theorists is the transport-cost differential over space. Although Ricardo concentrated more on fertility differences in his explanation of the land rent, von Thünen constructed a theory focusing on the transport-cost differentials across locations. For that, he used a very simple and elegant setting in which space is represented by a plain on which land is homogeneous in all respects except for a marketplace in which all transactions regarding final goods must occur. We will show that the price of land at any particular location reflects the proximity to the marketplace: the closer the marketplace, the higher the land rent.

More generally, the general principle holds that the distance to some specific places endowed with desirable attributes is the reason for having what is called a *differential land rent* ("differential" for differences in accessibility to some locations). Otherwise, how to explain why land rents are so high in cities? Indeed, in most habitable regions of the globe, the supply of land vastly exceeds the demand for land. Therefore, *absent the need for proximity, land should be* (almost) *a free good.*[2] This makes the case for proximity very strong. Several examples of this mechanism will be studied later in this chapter and in subsequent ones.

The location of the marketplace is supposed to be given, and the reasons for its existence will be analyzed in subsequent chapters. This marketplace is a major spatial inhomogeneity that allows one to obviate the difficulties raised by the spatial impossibility theorem. Specifically, we will see that very

[2] Land is also used for farming. However, the agricultural land rent is generally much lower than the urban land rent.

interesting results can be obtained with this model. In essence, the Thünen model shows how the existence of a center is sufficient for a competitive land market to structure the use of space by different activities. Not all transactions, however, need to occur at the market town. In particular, it seems reasonable to assume that intermediate inputs are traded on a local basis instead of being shipped to the marketplace. Therefore, we will extend the basic model by integrating intermediate goods, which are also produced from land but locally traded. This will allow us to shed light on the impact that technological linkages may have on the spatial distribution of activities under perfect competition. This is to be contrasted with the quadratic assignment problem in which technological linkages prevent the existence of a competitive equilibrium. The reason for this difference in results is that production activities are now assumed to be perfectly divisible, whereas the demand for the final goods comes from the marketplace, the location of which is fixed and given.

Our purpose in this chapter is not to provide a comprehensive survey of what has been accomplished in the large body of land use theory. Instead, we have chosen to focus on the main principles underlying Thünen's analysis. To this end, we discuss in Section 3.2.1 the properties of a simple, but suggestive, model formulated within the general competitive equilibrium framework. Specifically, we assume that (1) all agents are price-takers, (2) producers operate under constant returns to scale, and (3) there is free entry in each type of activity. The assumption of a competitive land market can be justified on the grounds that land in a small neighborhood of any location belonging to a continuous space is highly substitutable, thus making the competitive process for land very fierce.

However, because our main concern is to determine which agent occupies a particular location, it appears to be convenient, both here and in subsequent chapters in which we work with a land market, to determine the land use equilibrium from the *bid rent function* suggested by Thünen. The concept of bid rent function is probably what makes Thünen's analysis of land use so original and powerful. In a sense, it rests on the idea that land at a particular location corresponds to a single commodity whose price cannot be obtained by the textbook interplay between a large number of sellers and buyers, for as Alonso (1964, 41) put it, "land as space is a homogeneous good and land at a location is a continuously differentiated good."

Having said that, our aim is to find what kind of spatial distribution may arise in equilibrium as well as the features of the land rent profile sustaining such a distribution. Though the model is very simple, it shows that the spatial heterogeneity generated by a preexisting center is sufficient to obviate the negative conclusion of the spatial impossibility theorem. Two extensions of the basic model will be considered, namely, the introduction of intermediate goods (Section 3.2.2) and the possibility of substitution between land and labor in production (Section 3.2.3).

In Section 3.3, we will continue our exploration of the Thünian model by studying its applications to the formation of the urban land rent and the residential distribution of housing within a monocentric city. In this case, as suggested by Isard (1956, chap. 8) and formally developed by Alonso (1964), the Thünian town is reinterpreted as the city center (or central business district) to which individuals must commute in order to work, whereas housing is developed in the surrounding area. Our main focus here will be on the tension between the desire for space and the desire to commute less. We will see that this simple model provides a set of results consistent with the prominent feature of urban structures (Section 3.3.1). In particular, it explains the decrease in the urban land rent with distance away from the city center as well as the fall in the population density as one moves away from the center. As in the Thünian model, the city center plays a key role in the emergence of such a residential structure. Some comparative statics analysis is then performed on the residential equilibrium (Section 3.3.2). This analysis reveals several tendencies that agree with the main stylized facts suggested by urban economic history.[3] Among others, we note a spreading of the residential area corresponding to *suburbanization* when consumers get richer and commuting costs become lower, thus providing an explanation for what has been observed in many modern cities. We go on by showing that the market city is efficient in the absence of spatial externalities such as congestion in transport (Section 3.3.3).

In the foregoing analysis, the consumers are assumed to be identical in preferences and incomes. We go one step further by studying how the residential structure is affected when consumers are differentiated by their income (Section 3.3.4) and demonstrate that high-income consumers tend to settle far from the city center, which is left to the low-income ones. Such a pattern is commonly observed in North America, but not in Europe where the income gradient often slops downward as the distance to the city center increases.

[3] In this respect, the books by Bairoch (1985) and by Hohenberg and Lees (1985) offer both a great deal of relevant information.

We will see how historical and cultural amenities located at the city center may reverse the social stratification within the city. A similar setting will also allow us to shed light on the emergence of mixed urban-rural patterns (Section 3.3.6).

Finally, following the tradition of mainstream urban economics, we have assumed throughout this chapter a continuum of locations and consumers, thus working with a model in which *all* the unknowns are described by density functions. We show how the basic model of urban economics can be related to that of a city with a finite number of consumers located in a continuous urban space (Section 3.3.7). We conclude in Section 3.4 with a brief discussion of alternative, but related, urban models.

2 THE LOCATION OF DIVISIBLE ACTIVITIES

2.1 The Basic Model

By allocating an acre of land near the town to some crop, the costs of delivering all other crops are indirectly affected as they are forced to be grown farther away. Hence, determining which crops to grow where is not an easy task, thus making the work of Thünen very original. Though fairly abstract for the time, his treatment of the land use problem was not mathematical. One had to wait for the work of Launhardt ([1885], 1993, chap. 30) to have a formal treatment of his ideas in the special case of two crops. The first model dealing with an arbitrary number of crops is due to Dunn (1954).

The model is based on the following premises. There is a town located at the center of a featureless plain. All the products of various agricultural activities established in the surrounding area are to be traded there. The state formed by the town and its hinterland has no economic connections with the rest of the world; it is thus referred to as an *isolated state*. This isolated state is formally described by a large set of the Euclidean plane in which the town, treated as a point, is at the origin of the plane, whereas the distance from any point to the town is measured by the Euclidean distance. Each location r is identified by its distance r to the town.[4]

[4] In general, a point is described by its radius and its angle, but we may omit the angle because space is featureless around the city.

There are n activities, each producing a different agricultural good, or crop, denoted $i = 1, ..., n$. One may think of an activity as a set of farmers selling the same crop and using the same technology. The production of one unit of good i requires only the use of a_i units of land, where a_i is a positive constant independent of location, so that the technology of activity i exhibits constant returns to scale.[5] Consequently, if a unit of land at distance r is allocated to activity i, the corresponding production $q_i(r)$ of good i is given by

$$q_i(r) = \frac{1}{a_i}. \tag{1}$$

The density of land at each location is unity, and thus land density at distance r equals $2\pi r$.

Inasmuch as our focus is on land use, we put aside the determination of the prices of the agricultural goods in the town, which are supposed to be given and constant. Specifically, good i is sold at price p_i in the town to which it is shipped from its production place at a constant transport cost t_i per unit of good i and unit of distance. In other words, the product and transport markets are perfectly competitive.[6]

There is a perfectly competitive land market at every location in space, and the opportunity cost of land is assumed to be zero. However, as observed in the introduction, it is convenient to think that land at any point is allocated to an activity according to a bidding process in which the producer offering the highest bid secures the corresponding lot. In this regard, Thünen imagined a process in which each farmer makes an offer based on the surplus he can generate by using one unit of land available at any particular location. Because land is the only input and goods must be shipped to the market town, it should be clear that this surplus is given by $(p_i - t_i r)/a_i$. It

[5] We treat here a unit of land as a given combination of land and labor. Alternatively, we may consider that the price p_i introduced below represents the crop i's price net of all input-costs other than land rent. The cost of labor is explicitly accounted for in Section 3.2.3.

[6] Note, in passing, that Thünen used a more general specification of the transport cost involving two components. The first component corresponds to a monetary cost proportional to the quantity shipped and the distance covered (like ours), whereas the second is given by a fraction of the initial shipment's melting during the transport. For example, Thünen supposed that the cost of shipping grain consists partially of the grain consumed on the way by the horses pulling the load. This anticipates the iceberg cost proposed by Samuelson (1954a) and used in new economic geography (see Chapter 9).

varies with the activity but also with the location. Each activity i can then be characterized by a bid rent $\Psi_i(r)$ which is defined by the surplus per unit of land of any producer of good i at location r. Specifically, the *bid rent* of activity i at location r is here defined as follows:

$$\Psi_i(r) \equiv (p_i - t_i r)/a_i. \tag{2}$$

Since farmers are rational, they maximize profit per land unit. Being price-takers, the profit $\pi_i(r)$ made by a farmer in activity i per unit of land at location r is given by

$$\pi_i(r) = (p_i - t_i r)q_i(r) - R(r) = \Psi_i(r) - R(r)$$

using (1) and (2), where $R(r)$ is the rent per unit of land prevailing at distance r. Hence, a farmer's bid depends on both the transport rate of his output and the amount of land needed to produce one unit of the good. Since farmers compete for land until their profits are zero, the bid rent of those located at distance r coincides with the market land rent.

In the present setting, a competitive equilibrium is defined by a land rent function and by the areas in which each activity is undertaken such that no producer finds it profitable to change the location of its activity at the prevailing land rent. Because returns to scale are constant, it follows that any farmer with a positive output earns zero profits, whereas the equilibrium land rent cannot be negative. Consequently, (2) implies that the equilibrium land rent is such that

$$R^*(r) \equiv \max\left\{\max_{i=1,...,n} \Psi_i(r), 0\right\} = \max\left\{\max_{i=1,...,n} (p_i - t_i r)/a_i, 0\right\} \tag{3}$$

so that the *land rent function* $R^*(.)$ emerges as the upper envelope of the bid rent functions $\Psi_i(.)$. In other words, at the end of the bidding, each location is occupied by the agent who is able to offer the highest bid.[7]

[7] If the number of locations were finite, the land rent would be given by the outcome of an English auction in which the commodity is sold at the second highest reservation price. When the distance between adjacent locations along any ray goes to zero, the second highest reservation price tends to the highest reservation price at each location as given by Proposition 3.1 (Asami 1990). However, we must stress that, in more general settings in which the land use pattern is determined together with prices – wages or utility levels, say – this is no longer true. In such contexts, it is not clear how the bid rent function may emerge from a standard auctioning process.

Each bid rent function being decreasing and linear in distance, we may conclude as follows:

Proposition 1 *The equilibrium land rent function is the upper envelope of all bid rent functions, and each crop is raised where its bid rent equals the equilibrium land rent. If the transport cost function is linear in distance, then the equilibrium land rent is decreasing, piecewise linear and convex.*

As suggested in the introduction, it appears that the land rent is given by the *differential* surplus corresponding to the resources saved in transport by the most profitable activity relative to the zero surplus obtained at the extensive margin of land use. It even turns out that, for each activity, land rent is equal to the saving in transport cost. This strict relationship should not be overemphasized, however, because it depends on the assumption of fixed technological coefficients (see Section 3.2.3). An illustration of a land rent profile in the case of three activities is provided in Figure 3.1 where each linear segment of the land rent represents the bid rent of the corresponding crop.

Figure 3.1: The land rent profile and von Thünen rings when $n = 3$

It follows from Proposition 3.1 that, in equilibrium, crops are distributed according to the famous pattern of concentric rings centered at the market town, each of them being specialized in one crop. Then, the crops, if they are raised, are ordered by the distance from the town in such a way that the crop having the steepest bid rent function locates nearest to the town, the crop with the second steepest locates in the next ring, and so on. Hence, it is not true that zones near the market town are necessarily locations of intensive type of land use or are appropriated by activities producing transport costly goods. Instead, as one moves away form the center, it is the activity with the *steepest cost gradient*, defined here by the ratio t_i/a_i, that outbids the remaining activities, and secures the corresponding location. For example, if the activities use more or less the same amount of land per unit of output, the hard-to-transport goods, typically because they are perishable, are produced close to the market town, whereas the easier-to-ship goods are produced farther away from their consumption place. Conversely, if the transport rates are about the same across goods, land-intensive activities are located close to the market town, whereas land-extensive activities are developed far from the center.

Proposition 3.1 implies several other things. First, all activities are distributed around the center which, therefore, appears to be the pivotal element in the spatial organization of production. Second, each location is specialized and activities are spatially segregated within rings of land. However, we will see that, in other contexts (see Section 3.2.2), integrated configurations in which different activities are undertaken at the same locations may also arise in equilibrium. Last, any activity k such that

$$\Psi_k(r) < R^*(r) \qquad \text{for all } r \geq 0$$

has a zero output in equilibrium because it is unable to generate a surplus large enough to outbid the other activities anywhere in the plane.

Having this in mind, the following digression is in order. The market town may be viewed as the city-port of a small open economy. Once goods are gathered in the city-port, they may be sold within the country or exported to the rest of the world. In this case, the set of crops is primarily determined by the goods' prices that prevail on the international marketplace. Consequently, if the prices of the exported goods are sufficiently high, the goods consumed by the local population only may be no longer be available because they generate too low surpluses. In other words, trade seems to be harmful to the local population since the inhabitants must pay higher prices to import the agricultural products they need. This argument must be qualified, however. As will be seen below, the market distribution of crops generates the highest social surplus. Hence, the population is potentially better-off under trade because the surplus net of the costs of the imported goods is higher than the surplus made when the imported goods are locally produced. The key-issue is the structure of property rights. If land is collectively owned, people are better-off. On the contrary, if the land rent is captured by absentee landlords or by an unproductive elite, people are worse-off. Therefore, trade per se is not harmful to the local population. What matters for the well-being of the local population is the ownership structure of land.

For notational simplicity, we assume from now on that all activities have a positive output in equilibrium. Without loss of generality, we can re-index the activities in decreasing order of the slope (in absolute value) of their bid rent functions:

$$t_1/a_1 \geq ... \geq t_n/a_n.$$

We now show how the land use equilibrium pattern can be determined by using the bid rent function. We now show how the land use equilibrium pattern can be determined by using the bid rent function. Because activity 1 generates the highest surplus in the immediate vicinity of the town, it uses a disk of land (that is, a ring with a zero inner radius) whose radius r_1^* must satisfy

$$\Psi_1(r_1^*) = \Psi_2(r_1^*)$$

that is,

$$r_1^* = \frac{p_1/a_1 - p_2/a_2}{t_1/a_1 - t_2/a_2}$$

beyond which activity 2 is undertaken because its surplus becomes higher than that of activity 1. Similarly, activity i $(= 2, ..., n-1)$ will occupy a ring whose inner radius r_{i-1}^* is such that

$$\Psi_{i-1}(r_{i-1}^*) = \Psi_i(r_{i-1}^*)$$

whereas the outer radius r_i^* is the unique solution to

$$\Psi_i(r_i^*) = \Psi_{i+1}(r_i^*)$$

because the two bid rents are to be equal along the border between two adjacent rings. Solving this equation yields

$$r_i^* = \frac{p_i/a_i - p_{i+1}/a_{i+1}}{t_i/a_i - t_{i+1}/a_{i+1}}.$$

Finally, the external margin of land use is endogenously determined at the distance r_n^* from the market town at which

$$\Psi_n(r_n^*) = 0$$

because the opportunity cost of land is assumed to be zero:

$$r_n^* = p_n/t_n$$

so that land is used only within a bounded disk whose radius is given by r_n^*. Beyond this distance stands von Thünen's wilderness.

Since the equilibrium is competitive and there are no externalities, one expects the pattern of concentric rings to be socially optimal. That is, any other pattern in terms of size and shape would result in a lower *social surplus* S defined as the sum of crop surpluses minus transport costs:

$$S \equiv \sum_{i=1}^{n} p_i Q_i - \sum_{i=1}^{n} T_i \tag{4}$$

where Q_i is the output of activity i, and T_i is the corresponding transportation cost. Given (3), it is readily verified that the social surplus is here identical to the *aggregate land rent.*

Let $\theta_i(r) \geq 0$ denote the proportion of land used by activity i at distance r ($\sum_i \theta_i(r) \leq 1$). Then, because $2\pi r$ units of land are available at distance r, we have

$$Q_i = \int_0^\infty \theta_i(r) 2\pi r / a_i dr$$

and

$$T_i = \int_0^\infty [\theta_i(r) 2\pi r / a_i] t_i r dr.$$

Substituting Q_i and T_i into (4) and using (2), we obtain

$$S = 2\pi \int_0^\infty \left[\sum_{i=1}^{n} \theta_i(r) \Psi_i(r) \right] r dr$$

Maximizing S with respect to $\theta_i(.)$ is therefore equivalent to maximizing the bracketed term at each location r with respect to $\theta_i(r)$ subject to $\sum \theta_i(r) \leq 1$. Clearly, activity i is carried out at distance r if and only if $\Psi_i(r)$ is positive and the maximum of all bid rents. Therefore, both the optimum land use and market outcome are identical in the Thünen model, and both result in identical concentric rings.

Remark. It is worth mentioning that Koopmans and Beckmann (1957) also studied the *linear* assignment problem in which firm i receives a revenue from its sales to the rest of the world, which is location-specific. In this setting, firms do not exchange goods directly. Instead, outputs and inputs are shipped to some preexisting marketplaces where they are sold and bought, as in the von Thünen model. By relaxing the integer constraints on firms, the linear assignment problem can be expressed as a linear program. von Neuman has showed that the solution of this linear program is given by integer numbers, which means that each firm is assigned to a single location. Since the shadow prices generated by the dual of this program are location-specific, these prices have the nature of land rents. Thus, a competitive equilibrium exists and the optimal solution may be decentralized through a competitive land market.

The preceding analysis can be readily extended to the case of several production factors if production functions are of the fixed coefficient variety and if the return of each factor other than land is the same across locations. The case of a neoclassical technology is more complex and will be studied in Section 3.2.3.

The Thünen model can be closed by assuming that all agricultural activities need both land and labor while a $(n+1)$th manufactured good is produced in town by using labor alone, typically under the form of craftsmanship; such a specialization of tasks reflects the traditional division of labor between cities and the countryside. Workers are perfectly mobile between sectors and landlords reside in town; they all have identical (homothetic) preferences defined over the $(n+1)$ goods. The solution to such a general spatial equilibrium model, in which the real wage common to all workers as well as the prices of agricultural and manufactured goods are endogenous, has been studied by Samuelson (1983) when $n = 2$ and by Nerlove and Sadka (1991) when $n = 1$.

Thus, what remains to be explained is when a market town, which imports agricultural goods from and exports manufactured goods to its rural hinterland, emerges as an equilibrium outcome. To put it differently, *what binds together manufacturing firms and workers within a city?* This question has been at the heart of geographical economics for decades. In chapter 10, we will answer this question by using a more general research strategy.

2.2 Technological Linkages and the Location of Activities

So far we have assumed that any produced good is shipped to the market town in which it is consumed. A well-known difficulty encountered in economics is to account for the existence of intermediate goods. It is interesting to figure out what the ring-shaped pattern obtained in the Thünian model becomes when some goods are used as inputs in the production of other goods. To the best of our knowledge, this problem has been first modeled by Mills (1970; 1972a, chap. 5) and extended further by Goldstein and Moses (1975). These authors assumed that intra-area shipments go by the shortest route and need not be shipped through the town.

The main change in the spatial organization of production is that several goods may be produced simultaneously at the same location instead of being produced in separated locations, as in the preceding section. To illustrate the working of such an economy, we adopt a slightly modified version of Mills by assuming that only two goods are involved, good 2 being used only as an input for producing good 1, which is itself shipped to the market town for being sold at a given price p_1. We will study this particular model in detail because it will allow us to see how all the equilibrium conditions interact to determine the equilibrium configuration and why the assumption of complete markets is needed.

As before, the production of one unit of good i requires a fixed amount of land a_i. However, producing one unit of good 1 requires also b units of good 2. Without loss of generality, the unit of good 2 is chosen for $b = 1$. Hence, it must be that $Q_1 = Q_2$.

It is worth noting that the equilibrium distance to the external land margin r_2 depends on the total production of good 1 but not on the way land is allocated between the two activities. Indeed, we have

$$a_1Q_1 + a_2Q_2 = (a_1 + a_2)Q_1 = \pi r_2^2$$

so that

$$r_2 = \left[(a_1 + a_2)Q_1/\pi\right]^{1/2}.$$

An equilibrium configuration arises when no producer wants to change the location of his activity at the prevailing land rent and factor prices and when the market clearing conditions for the intermediate product hold. Because the model is linear, we may focus on the following two polar configurations: the integrated one, where both activities are undertaken together at each location, and the segregated one, where the two activities are separated as in Section 3.2.1. The spatial price equilibrium conditions imply that it is never profitable to transport good 2 when the configuration is integrated; when the configuration is segregated, they say that the price of good 2 at any location where good 1 is produced is equal to the cost for one unit of good 2 to be available at the border between the two areas plus the transport cost from the border point to the production point.

To identify the conditions under which each configuration emerges as an equilibrium, it is again useful to work with the bid rent function associated with each activity. If $p_2^*(r)$ stands for the equilibrium price of good 2 at r, the surplus per unit of land (or the bid rent) of activity i at each r is defined as follows:

$$\Psi_1(r) = \frac{p_1 - t_1 r - p_2^*(r)}{a_1} \tag{5}$$

$$\Psi_2(r) = \frac{p_2^*(r)}{a_2}. \tag{6}$$

First, consider an integrated configuration. For such a configuration to emerge, the two activities must have the *same* bid rent at each $r \leq r_2$, as illustrated in Figure 3.2. That is,

$$\frac{p_1 - t_1 r - p_2^*(r)}{a_1} = \frac{p_2^*(r)}{a_2}$$

or

$$p_2^*(r) = \frac{a_2}{a_1 + a_2}(p_1 - t_1 r). \tag{7}$$

Setting $\Psi_2(r) = 0$ (or $p_2^*(r) = 0$) at r_2, we obtain the fringe distance as follows:

$$r_2^* = p_1/t_1.$$

The integrated configuration is an equilibrium if and only if shipping good 2 must never be profitable. Since (7) is linear in distance, this amounts to

$$t_2 \geq \left| \frac{dp_2(r)}{dr} \right| = \frac{a_2 t_1}{a_1 + a_2}$$

or

$$\frac{t_2}{t_1} \geq \frac{a_2}{a_1 + a_2}. \tag{8}$$

This condition means that, given the relative intensity of land use in producing the two goods, shipping one unit of good 2 is more costly than shipping one unit of good 1, and thus it is preferable to save on the transport of 2 than on the transport of 1.

Figure 3.2 : The rent profile for the integrated configuration
Figure 3.3 : The rent profile for the segregated configuration

The case of a segregated configuration is a bit more involved. Assume as in Figure 3.3 that good 1 is produced up to r_1, whereas good 2 is produced beyond r_1 up to r_2. Since the market for good 2 is competitive, everything works as if there was a marketplace for good 2 located in town, where this good is sold at some equilibrium price p_2^*. When good 2 is used at r, we have

$$p_2^*(r) = p_2^* - t_2 r \tag{9}$$

Substituting (9) into (5) and (6) yields

$$\Psi_1(r) = \frac{p_1 - p_2^* - (t_1 - t_2)r}{a_1}$$

$$\Psi_2(r) = \frac{p_2^* - t_2 r}{a_2}.$$

The three unknowns, p_2^*, r_1, and r_2, can be determined by using the following equilibrium conditions. First, the two activities have the same bid rent at r_1:

$$\frac{p_1 - p_2^* - (t_1 - t_2)r_1}{a_1} = \frac{p_2^* - t_2 r_1}{a_2}.$$

Second, the bid rent of activity 2 is zero at r_2:

$$r_2 = p_2^*/t_2.$$

Third, $Q_1 = Q_2$ implies

$$\frac{\pi r_1^2}{a_1} = \frac{\pi(r_2^2 - r_1^2)}{a_2}$$

and thus

$$r_2 = \left(\frac{a_1 + a_2}{a_1}\right)^{1/2} r_1 \equiv k r_1$$

where $k > 1$. The three conditions above yields

$$r_1^* = \frac{a_2 p_1}{(k-1)(a_1 + a_2)t_2 + a_2 t_1}$$

$$r_2^* = \frac{a_2 k p_1}{(k-1)(a_1+a_2)t_2 + a_2 t_1}$$

and

$$p_2^* = k t_2 r_1^*.$$

For the segregated configuration to be an equilibrium, as shown in Figure 3.3, the bid rent curve of crop 1 must intersect that of crop 2 from above at distance r_1^*, thus implying

$$-\frac{d\Psi_1(r)}{dr} \geq -\frac{d\Psi_2(r)}{dr}.$$

This amounts to

$$\frac{t_1 - t_2}{a_1} \geq \frac{t_2}{a_1}$$

or

$$\frac{t_2}{t_1} \leq \frac{a_2}{a_1 + a_2}. \tag{10}$$

To sum up, we have shown:

Proposition 2 *(i) If*

$$\frac{t_2}{t_1} \geq \frac{a_2}{a_1 + a_2}$$

holds, then the integrated configuration is an equilibrium.
(ii) If

$$\frac{t_2}{t_1} \leq \frac{a_2}{a_1 + a_2}$$

holds, then the segregated configuration is an equilibrium.

Though an equilibrium always exists, it involves positive transport costs. This does not contradict the spatial impossibility theorem because the existence of a center turns out to be a spatial inhomogeneity that facilitates coordinating producers' decisions. In addition, activities are perfectly divisible, which means that they do not have a restricted number of addresses. Furthermore, the equilibrium may involve positive interactivity transport costs as in the quadratic assignment problem studied in Chapter 3. This is so when the cost t_2 of shipping the intermediate good to the producers of good 1 is low relative to the cost t_1 of shipping the final product to the market town. In this case, the equilibrium involves specializing land in the production of good

1 in the vicinity of the center, whereas good 2 is produced farther away; the pattern of production is ring-shaped as in the Thünian model. Otherwise, the two activities are spatially integrated to save the interactivity transport costs because activities are perfectly divisible, unlike what was assumed in the quadratic assignment problem. Thus, in the presence of intermediate goods, the equilibrium does not necessarily involve spatial specialization.[8]

Consequently, when there are technological linkages, the type of spatial configuration emerging at the market solution varies with the relative value of the transportation rates. This has an important implication: The fall in transport costs observed since the beginning of the Industrial Revolution does not imply that activities become indifferent with respect to their location. Even though transport costs would decrease, what matters for the organization of space is their relative changes.

The set of equilibrium patterns becomes richer once we allow for a more general input-output structure and relax the assumption of an isolated state by permitting imports through the market town at given prices p_1 and p_2 (Goldstein and Moses, 1975). For example, when each activity uses the output of the other, if there are no imports, the inner ring is specialized in activity 1 whereas the outer ring involves integration: good 2 is produced for use in the first ring, but also for producing locally good 1 in the second ring, which, in turn, is used as a local input for producing good 2. When they compare their approach to the quadratic assignment model, Goldstein and Moses (1975, 77) are right when they claim:

> By setting up a model with two goods, and a marketing center we are able to reach an equilibrium with complete interdependence and positive transport costs.

We may thus safely conclude that the continuous approach to land use combined with the existence of a marketplace for the final goods lead to important results with nontrivial equilibria. Unfortunately, determining the market outcome becomes quickly intractable when the number of goods increases, due to the large number of possible special cases involved in characterizing equilibria. However, we have seen that there is never outward shipment of goods in equilibrium: either good 2 is consumed on the spot (as in the case of an integrated configuration) or transported toward the inner ring (as in the case of a segregated configuration). Schweizer and Varaiya

[8] See Chapter 6 for a similar result in a different context.

(1976a) have shown that, in the general case of n goods under any Leontief technology in which goods may be used both in the final and intermediate sectors, the equilibrium always involves one-way trade: goods are either shipped toward the marketplace or used locally.

As seen above, the work by Koopmans and Beckmann (1957) has been at the origin of a long-standing debate about the (im)possibility of decentralizing the optimal configuration in a spatial economy. Of course, for this question to be addressed properly, one must work within a framework in which nontrivial competitive equilibria exist. In this perspective, Proposition 3.2 offers an interesting starting point. Furthermore, Mills (1970; 1972a, chap. 5) also showed that, in the model discussed above, the integrated solution is socially optimal if and only if (8) is verified, that is, when it pays to save on the transport of the intermediate good despite the need of shipping the final good to the center. On the contrary, when (10) holds, it is the segregated configuration that is socially desirable because it now pays to economize on the cost of shipping the final product. In both cases, the optimum can be sustained as an equilibrium. This turns out to be a fairly general property: Schweizer and Varaiya (1976a) have shown that, in a monocentric economy, the optimal configuration can always be sustained by a decreasing and convex land rent in the general case of n goods. Therefore, in a monocentric economy with divisible activities and technological linkages, the second welfare theorem holds.

Accordingly, the existence of intermediate goods need not prevent the existence of a competitive equilibrium when activities are perfectly divisible and when there exists a single market place for some goods. In addition, the analysis of Mills reveals that any equilibrium is an optimum, that is, the first welfare theorem also holds (Goldstein and Moses, 1975). Again this seems to be fairly robust in the case of divisible activities, though a general result comparable to Schweizer and Varaiya is missing. The divisibility of activities makes the accessibility of an activity to the others potentially free since an integrated configuration is always feasible, whereas the existence of a single marketplace is a spatial inhomogeneity that plays the role of a coordination device among producers. In such a context, there is no market failure. We will see an example of such a result in Section 3.3.3.

2.3 The Case of a Neoclassical Technology

Even though Thünen is considered the founder of marginalism, his model still belongs to the realm of classical economics to the extent that it assumes fixed technological coefficients. A more modern approach is obtained once substitution between land and labor, say, is allowed. This problem was tackled by Beckmann (1972b), who considered the case of a neoclassical Cobb–Douglas production function, but more general well-behaved production functions could be similarly considered. Here we present a slightly more general analysis of this problem in that the parameter of this function may vary across activities. We assume that the assumptions of Section 3.2.1 are still valid, but (1) is now replaced by

$$q_i(r) = f[x_i(r)] = [x_i(r)]^{\alpha_i}$$

where $x_i(r)$ denotes the quantity of labor units used per unit of land, whereas $q_i(r)$ is the output of good i per unit of land. In this expression, $0 < \alpha_i < 1$ stands for the substitution parameter between land and labor for good i. Hence the marginal productivity of labor is positive and decreasing; the marginal productivity of land, given by $f(x_i) - x_i f'(x_i)$, is also positive and decreasing.

The profits $\pi_i(r)$ per unit of land earned by a producer at location r are then given by

$$\pi_i(r) = (p_i - t_i r)q_i - wx_i - R(r) \tag{11}$$

where w is the wage rate, which is assumed to be given and constant across locations. Therefore, the corresponding profit-maximizing demand for labor is

$$x_i^*(r) = \left[\frac{\alpha_i(p_i - t_i r)}{w}\right]^{\frac{1}{1-\alpha_i}} \qquad \text{for } r \leq \frac{p_i}{t_i}. \tag{12}$$

Accordingly, for each activity, less and less labor is used as one moves away from the market town so that the equilibrium output is decreasing and continuous in the distance to the market town. Plugging (12) into (11) and setting $\pi_i(r) = 0$ allows us to determine the maximum surplus that activity i may generate at location r. Consequently, the bid rent function associated with this activity is now defined by

$$\Psi_i(r) = (1 - \alpha_i)\,(\alpha_i/w)^{\beta_i}\,(p_i - t_i r)^{1+\beta_i} \qquad \text{for } r \leq \frac{p_i}{t_i}$$

where $\beta_i \equiv \alpha_i/(1-\alpha_i) > 0$. Hence, each bit rent function is decreasing and strictly convex in distance.

Without loss of generality, let $p_1/t_1 \leq ... \leq p_n/t_n$. Using the same argument as in Section 3.2.1, it may be shown that the equilibrium land rent is now given by

$$\begin{aligned} R^*(r) &\equiv \max\left\{\max_{i=1,\ldots,n} \Psi_i(r), 0\right\} \\ &= \max\left\{\max_{i=j,\ldots,n} (1-\alpha_i)\left(\alpha_i/w\right)^{\beta_i}\left(p_i - t_i r\right)^{1+\beta_i}, 0\right\} \qquad \text{for } p_{j-1}/t_{j-1} \leq r \leq p_j/t_j \end{aligned}$$

and thus:

Proposition 3 *If production is described by a homogeneous linear Cobb-Douglas function and if the wage rate is constant across locations, then the equilibrium land rent is decreasing and strictly convex in distance to the market town.*

Hence, using a neoclassical production function does not affect the general pattern of location, which is still described by a set of specialized concentric rings, whereas the land rent keeps the same decreasing and convex shape, as in the Thünen model.

However, the simple and elegant condition describing the sequence of land use zones in the Thünen model does not hold anymore: the same crop may be raised within two different rings because Ψ_i and Ψ_j may intersect more than once. Furthermore, the employment level may not be a continuous and decreasing function across activities. We have seen that this function is continuous and decreasing within each ring but this does not necessarily hold at the border between two adjacent rings. Indeed, the equilibrium conditions imply that, at any distance r where activity i is undertaken, the land rent equals the marginal productivity of land whereas the wage equals the marginal productivity of labor, that is:

$$R(r) = (1-\alpha_i)[x_i^*(r)]^{\alpha_i}(p_i - t_i r)$$

as well as

$$w = \alpha_i[x_i^*(r)]^{\alpha_i - 1}(p_i - t_i r).$$

Taking the ratio of these two expressions yields

$$\frac{R(r)}{w} = \frac{x_i^*(r)}{\beta_i}.$$

Since, at the border r_i^* between the i-th and $(i+1)$th rings, the same relationship holds for activity $i+1$ and since $R(r)/w$ is the same, it must be that

$$\frac{x_i^*(r_i^*)}{\beta_i} = \frac{x_{i+1}^*(r_i^*)}{\beta_{i+1}}.$$

Hence, the employment level is continuous across activities ($x_i^*(r_i^*) = x_{i+1}^*(r_i^*)$) if and only if the coefficients β_i are the same for all activities, that is, the production functions are identical for all activities. In this case, the equilibrium employment is a continuous and decreasing function of the distance to the market town across locations and activities.

On the other hand, if the coefficients α_i differ across activities there is a discontinuity in the employment level at the border between two adjacent rings. Nevertheless, this input might still be decreasing. Let us check when this is so. For $x_i^*(r_i^*) > x_{i+1}^*(r_i^*)$ to hold, it must be that $\beta_i < \beta_{i+1}$, that is, $\alpha_i > \alpha_{i+1}$. Therefore, in equilibrium, the labor input is decreasing (but not continuous) provided that the locations of activities are ordered by decreasing order of the share of labor in the production of goods. There is no reason to expect this condition to be satisfied at the equilibrium configuration. Though the consumption of land remains specialized and ring-shaped, it therefore appears that the employment level may jump up or down when land use shifts from one activity to the next once substitution between land and labor is allowed.

Note, finally, that the inspection of the market land rent $R^*(r)$ reveals that, for any given activity, the decrease in the land value no longer fully compensates for the corresponding increase of the transport cost. The change in land price now induces a substitution from labor to land as one moves away from the market town, thus making this relationship more involved. We will return to this problem in Section 3.3.1.

2.4 Notes on the Literature

A lot of attention has been devoted to the possible re-switching of technologies as one moves away from the market town. The main results can be found in Schweizer and Varaiya (1976b) and Schweizer (1980). Another domain of application of the von Thünen lies (somewhat ironically) in the neo-Ricardian models of production considered by Scott (1977) and Huriot (1981).

3 THE URBAN LAND RENT

3.1 Residential Equilibrium in the Monocentric City

Two fundamental ideas lie at the heart of the economics of city structure: (i) people prefer shorter commuting trips to longer commuting trips and (ii) people prefer having more space than less space. Starting from these premises, the analysis of the internal structure of a monocentric city relies on the approach developed by Alonso (1964), Mills (1967), and Muth (1969), that is, *the households' trade-off between housing size and accessibility to the city center where jobs available.*

To illustrate how this trade-off works, we consider a city with a prespecified center, called the central business district (CBD), where all jobs are located. For simplicity, the CBD is treated as a point, and space is assumed to be homogeneous except for the distance to the CBD. In this context, the only spatial characteristic of a location is its distance from the CBD, and thus the model is essentially one-dimensional. Compared with the Thünian model presented in Section 3.2.1, it therefore appears that the CBD replaces the market town, whereas the land available for raising crops is now used for housing. The land market works as if each household were to compare possible locations and evaluate, for each location, the maximum rent per unit of surface it would be willing to pay to live there. Each plot is then occupied by the household offering the highest bid. Competition for housing near the CBD where jobs are located leads households to pay a land rent that varies inversely with the distance between homes and jobs. Or, to put it differently, people trade bigger plots against higher commuting costs. Despite its simplicity, we will see how the monocentric city model highlights a major principle of geographical economics, that is, *the land rent reflects the accessibility to a specific place endowed with features that economic agents value.*

Consider a continuum N of identical workers/consumers commuting directly to the CBD where they earn a given, fixed income Y. Each consumer has a utility U depending on the quantity z of a composite good, which is available everywhere at a price equal to 1, and the lot size s of housing.[9] It is

[9] In order to focus on lot size and population density changes within the city, we use a simple utility with two arguments, z and s. However, the model can readily be extended to the case of several consumption goods as well as to nonland input for housing (Fujita, 1989, 44).

assumed that U is strictly increasing in each good, twice continuously differentiable, strictly quasi-concave while both z and s are essential goods (every indifference curve has each axis as an asymptote). Furthermore, the lot size s is assumed to be a normal good. If a consumer is located at a distance r from the CBD, his budget constraint is then given by $z + R(r)s + T(r) = Y$ where $R(r)$ is the rent per unit of land at r and $T(r)$ is the commuting costs at r. We suppose that there is no congestion in commuting while $T(r)$ is strictly increasing in distance and $0 \leq T(0) < Y < T(\infty)$.

The residential problem of the consumer can then be expressed as follows:

$$\max_{r,z,s} U(z, s) \qquad \text{s.t.} \quad z + sR(r) = Y - T(r) \tag{13}$$

where $Y - T(r)$ is the net income at r. The only difference from the standard consumer problem is that here the consumer must also choose a residential location $r \geq 0$, which affects the land rent he pays, his commuting cost and his consumption bundle. It should be clear that this problem encapsulates the trade-off between accessibility to the CBD, measured by $T(r)$, and the land consumption, measured by s.

Because consumers are identical in terms of preferences and income, at the *residential equilibrium* they reach the same utility level u^* regardless of their location. Observe the difference with the bid rent defined by (2) in the Thünian model in which it is implicitly assumed that the equilibrium profit level of activity i is zero. By contrast, the equilibrium utility level u is endogenous here because there are no in- or out-migrations. This makes the land market across locations interdependent. Yet, as in the Thünian model, we define the bid rent function $\Psi(r, u)$ of a consumer as the maximum rent per unit of land that he is willing to pay at distance r while enjoying the utility level u.[10] Given (13), we have

$$\Psi[Y - T(r), u] \equiv \Psi(r, u) = \max_{z,s} \left\{ \frac{Y - T(r) - z}{s} \quad \text{s.t.} \quad U(z, s) = u \right\}. \tag{14}$$

Indeed, for the consumer residing at distance r and selecting the consumption bundle (z, s), $Y - T(r) - z$ is the money available for land payment, and thus $(Y - T(r) - z)/s$ represents the rent per unit of land at r. The bid

[10] The bid rent function approach followed here is essentially the same as the indirect utility function approach used by Solow (1973) and is, therefore, closely related to duality theory as developed in microeconomics.

rent $\Psi(r,u)$ is then obtained by choosing the utility-maximizing consumption bundle (z,s) subject to the constraint $U(z,s)=u$.[11]

Because U is strictly increasing in z, the equation $U(z,s)=u$ has a unique solution denoted by $Z(s,u)$. It is readily verified that the quantity $Z(s,u)$ of the composite good is strictly decreasing, strictly convex, strictly increasing in u, and such that $\lim_{s\to 0} Z(s,u)=\infty$. Consequently, (14) can be rewritten as follows:

$$\Psi(r,u)=\max_{s}\frac{Y-T(r)-Z(s,u)}{s}. \tag{15}$$

It follows from this expression that the equilibrium consumption bundle of a consumer located at r is obtained at the tangency point between the budget line whose slope equals $\Psi(r,u)$ and the indifference curve of level u in the positive orthant of the (z,s)-plane, as illustrated in Figure 3.4.

Figure 3.4: The equilibrium consumption bundle at r

For each r at which the net income $Y-T(r)$ is positive, the unique solution to (15) is denoted by $S(r,u)$.

The price of the composite good being 1, the indirect utility when the land rent is R and the net income I is denoted $V(R,I)$. By definition of the bid rent, we have the identity:

$$u\equiv V\left[\Psi(r,u),Y-T(r)\right]. \tag{16}$$

The land consumption $S(r,u)$ is given by the Marshallian demand $\widehat{s}(R,I)$. Because consumers have the same utility level across space, $\widehat{s}(R,I)$ is also equal to the Hicksian demand $\widetilde{s}(R,u)$:

$$S(r,u)\equiv\widehat{s}\left[\Psi(r,u),Y-T(r)\right]\equiv\widetilde{s}\left[\Psi(r,u),u\right]. \tag{17}$$

In other words, when a consumer changes his location r, his bid rent is adjusted for his utility level to remain the same.

We are now prepared to characterize the bid rent and the lot size functions. Differentiating (15) and applying the envelope theorem, we obtain

$$\frac{d\Psi(r,u)}{dr}=-\frac{T'(r)}{S(r,u)}<0 \tag{18}$$

[11] Because U is strictly quasi-concave, the utility-maximizing bundle (z,s) exists and is unique as long as the net income $Y-T(r)$ is positive.

because $T(r)$ is strictly increasing in r and $Z(s,u)$ strictly increasing in u. Thus, using (17) yields

$$\frac{dS(r,u)}{dr} = \frac{\partial \widetilde{s}}{\partial R}\frac{d\Psi(r,u)}{dr} = -\frac{\partial \widetilde{s}}{\partial R}\frac{T'(r)}{S(r,u)} > 0 \tag{19}$$

because the Hicksian demand for land is strictly decreasing in land price.

Similarly, we have

$$\frac{d\Psi(r,u)}{du} = -\frac{1}{S(r,u)}\frac{\partial Z(s,u)}{\partial u} < 0 \tag{20}$$

because $Z(s,u)$ is strictly increasing in u. In other words, households' willingness-to-pay for one unit of land decreases when a higher utility level is sustainable, e.g. when the income Y rises. Indeed, when households are richer, the income share devoted to commuting decreases, and thus proximity to the CBD matters less. Note, however, that

$$\frac{dS(r,u)}{du} = \frac{\partial \widetilde{s}}{\partial R}\frac{d\Psi(r,u)}{du} > 0 \tag{21}$$

which means that these households consume a larger land plot.

Thus, we have shown:

Proposition 4 *The bid rent function is continuously decreasing in both r and u (until it becomes zero). Furthermore, the lot size function is continuously increasing in both r and u.*

When $T(r)$ is linear or concave in distance, $\Psi(r,u)$ is strictly convex in r as shown by differentiating (18) with respect to r and using (19).

We now turn to the description of the equilibrium conditions for the monocentric city in which N homogeneous consumers have a given income Y earned at the CBD. Landowners are assumed to be absentee, and thus the land rent is not distributed to consumers. The equilibrium utility u^* is the maximum utility attainable in the city under the market land rent $R^*(r)$. Using (16), we obtain

$$u^* = \max_r V\left[R^*(r), Y - T(r)\right] \tag{22}$$

which is the common utility level in equilibrium.

If one differentiates (16) with respect to r and uses Roy's identity, the utility-maximizing choice of a location by a consumer at the residential equilibrium implies

$$S(r, u^*)\frac{dR^*(r)}{dr} + \frac{dT(r)}{dr} = 0. \tag{23}$$

That is, at the residential equilibrium, changes in *housing costs* evaluated at the utility-maximizing land consumption are balanced by the corresponding changes in *commuting costs*. In particular, when the lot size is fixed and normalized to 1 ($s = 1$), we get a "flat city" in which (23) becomes $dR^*(r)/dr + dT(r)/dr = 0$, and thus

$$R^*(r) + T(r) = \text{constant}. \tag{24}$$

In this case, as in the Thünian model, the shape of the land rent is the opposite of the shape of the commuting cost function, whereas the consumption of the composite good is the same across consumers. To put it differently, housing plus commuting costs are the same across the city. By contrast, when the lot size is variable, that mirror relationship ceases to hold. Unexpectedly, however, these two magnitudes remain closely related at the aggregate level: If commuting costs are linear in distance, then the aggregate differential land rent is just equal to total commuting costs when the city is linear, whereas it equals half the total commuting costs when the city is circular (Arnott 1979; 1981).

Let $n(r)$ be the consumer density at distance r in equilibrium. Then, we have

$$\begin{array}{lcll} R^*(r) & = & \Psi(r, u^*) & \text{if} \quad n(r) > 0 \\ R^*(r) & > & \Psi(r, u^*) & \text{if} \quad n(r) = 0. \end{array}$$

If it is assumed that land not occupied by consumers is used for agriculture yielding a constant rent $R_A \geq 0$, the city fringe arises at distance r^* such that

$$\Psi(r^*, u^*) = R_A. \tag{25}$$

The bid rent being decreasing in r by Proposition 3.4, the residential area is given by a disk centered at the CBD having radius r^*. As a consequence, the market land rent is given by

$$R^*(r) = \left\{ \begin{array}{lll} \Psi(r, u^*) & \text{for} & r \leq r^* \\ R_A & \text{for} & r \geq r^* \end{array} \right. \tag{26}$$

Because no land is vacant within the urban fringe, we must have

$$n(r) = 2\pi r / S(r, u^*) \qquad \text{for all } r \leq r^* \tag{27}$$

and thus the total population N within the urban area must satisfy

$$\int_0^{r^*} \frac{2\pi r}{S(r, u^*)} dr = N. \tag{28}$$

In summary, the residential equilibrium is described by $R^*(r)$, $n^*(r)$, u^* and r^* satisfying the conditions (25) through (28). Under the preceding assumptions made about preferences, income and commuting costs, the existence of a unique residential equilibrium can be shown to hold (Fujita 1989, Proposition 3.1).

Proposition 3.4 and (26) imply that within the urban area the market land rent is decreasing as one moves away from the CBD, which is a result that also holds in the Thünian model. Denoting the population density at r by $\delta(r) \equiv n^*(r)/2\pi r$, we see from (27) that

$$\delta(r) = 1/S(r, u^*).$$

We may then conclude from Proposition 3.4 that *the equilibrium population density is decreasing from the CBD to the urban fringe*, whereas the equilibrium land consumption simultaneously rises. In other words, consumers trade more (less) space for housing against a lower (higher) accessibility to the CBD in a way that allows them to reach the same highest utility level across locations. In monetary terms, a consumer paying a high (low) price for land bears low (high) commuting costs but the compensation is not necessarily exact because the consumption of the composite good also changes with r. Indeed, each consumer residing further away from the city center has a larger consumption of land and a smaller consumption of the composite good for the utility level to be the same across the city.[12] Accordingly, *space is sufficient to render heterogeneous consumers who are otherwise homogeneous.*

Therefore, the equilibrium city accommodating a population of N consumers is described by a circular area centered at the CBD. The consumer density as well as the land rent fall as the distance to the city center rises. This provides an explanation for the fairly general empirical fact that the

[12] As seen in the foregoing, there is exact compensation when the lot size is fixed.

population density is higher near the city center (where housing costs are high) than at the city outskirts (where housing costs are low). In addition, the size of the residential area depends on the opportunity cost of land but also on the number of consumers, their income, and the value of their commuting costs to the CBD. These relations will be used in Section 3.3.2 to explain another major fact about urban areas, namely suburbanization.

3.2 Comparative Statics of the Residential Equilibrium

We can now perform some comparative statics that will shed additional light on real world issues. First, an increase in the population size has fairly straightforward effects. Indeed, a rising population makes competition for land fiercer, which in turn leads to an increase in land rent everywhere and pushes the urban fringe outward. This corresponds to a well-documented fact stressed by economic historians. Examples include the growth of cities in Europe in the twelfth and nineteenth centuries as well as in North America and Japan in the twentieth century or since the 1960s in Third World countries. All were caused by demographic expansion and rural–urban migrations resulting from technological progress in agriculture, which freed some population from agricultural activity (Bairoch 1988, chaps. 10 and 14; Hohenberg and Lees 1995, chaps.????).[13] As a result, the inhabitants of a city facing an inflow of migrants bear higher housing and commuting costs, a force that limits the agglomeration of activities within a city. To see how this works, consider the simple case where the lot size is fixed. Using (24), we obtain $R^{*}(r) + T(r) = R_A + T(N/2)$, which is indeed an increasing function of the population size N.

We now investigate the impact of a rise in consumers' income Y. Using (25) and (28), we can readily verify that the residential area expands because the urban fringe moves outward. Although all consumers are clearly strictly better off, the impact on the land rent and the population density is less obvious. An increase in consumers' income raises demand for land everywhere. However, it also leads to a decrease in the relative value of commuting costs, thus making locations in the suburbs more desirable than before the income rise. Consequently, because enough land is available in the suburbs (recall

[13] The same force is still at work today. For example, Baum-Snow (2007) shows that one new highway passing through a US central city reduces its population by about 18 percent.

that the additional land available between r and $r + dr$ is $2\pi r dr$), a substantial segment of the population will move from the center to the suburbs. This will in turn decrease the land rent and the population density near the CBD but increase them in the suburbs. In other words, both the land rent and the population density become flatter. Because the locational decision of a consumer is governed by his net income, decreasing the commuting costs has exactly the same impact as increasing Y. We may then conclude that, since the development of modern transportation means (mass transportation and cars) that have followed the Industrial Revolution, income has increased and commuting costs have decreased, generating both suburbanization and a flattening of the urban population densities in many American and European cities (Bairoch 1988, chap. 19; Hohenberg and Lees 1995, chap.???).

Finally, consider an increase in the opportunity cost of land as measured by the agricultural land rent R_A. Using (25) and (28), one can show that the urban fringe shrinks, whereas the equilibrium utility level falls as R_A rises. Then, Proposition 3.4 implies that both the market land rent and consumer density are higher at any distance within the new urban fringe. Hence a higher opportunity cost of land leads to a more compact city with more consumers at each location paying a higher land rent. Increasing the opportunity cost of land therefore leads to more concentrated populations and less well-being for consumers, as suggested by the current situation in many cities in Japan or other countries in East Asia. A high opportunity cost for land may be due to the relative scarcity of land, but it may also find its origin in public policies that maintain the prices of agricultural products far above the international level.[14] This also explains why, for centuries, the spatial extension of towns was limited by returns in agricultural activities as well as by the transport means available to ship produce (Bairoch 1988,

[14]According to Ohmae (1995, 48), "within a 50-kilometer radius of Tokyo, 65 percent of land – nearly 330,000 hectares of some of the most expensive property in the world – is devoted to widely inefficient agriculture. If only one quarter of this land were sold for private housing, Tokyo-area families would be able to afford 120 to 150 square meters of living space, instead of today's average of 88 square meters. Moreover, cheaper – and more available – land would cut the cost of essential public work like providing better sewage, removing traffic bottlenecks, and double-tracking commuter trains." In the same spirit, restrictions on building height are also restrictions on the supply of living space. For example, Glaeser (2011) provides ample evidence of the perverse effects generated by the web of regulation prevailing in Manhattan that result in artificially high housing prices, which benefit the incumbents at the expense of new comers who are relegated in distant neighborhoods.

chap. 1).

3.3 Efficiency of the Residential Equilibrium

It remains to discuss the efficiency of the residential equilibrium. Because this equilibrium is competitive (consumers are price takers) and no externalities are involved, the first welfare theorem suggests that the equilibrium is efficient. However, we have here a continuum of commodities (land), and thus we need a more specific argument.

It is well known in urban economics that using a utilitarian welfare function leads to the unequal treatment of equals (Mirrlees 1972), whereas equals are equally treated in equilibrium. Such a difference is unexpected, and one might think that competition for space leads to strong social inefficiencies even though our economy is competitive. However, Wildasin (1986a) has shown that this pseudo-paradox arises because the marginal utility of income is different across consumers at different locations. Using a utilitarian approach is therefore unjustified. This fact invites us to consider an alternative approach in which the utility level is fixed across identical consumers.

Assume, then, that all consumers achieve the equilibrium utility level u^* and check whether there exists another feasible allocation $(n(r), z(r), s(r); 0 \leq r \leq \widehat{r})$ that sustains u^* and reduces the social cost C. Note that such an allocation maximizes a Rawlsian welfare function (maximizing the minimum utility level in the economy) when the social planner cannot use lump-sum transfers. This has major implications that will be discussed in subsequent chapters.

In our model, the social cost for N consumers to enjoy the utility level u^* is obtained by summing the commuting costs, the composite good cost and the opportunity land cost borne by society for this to be possible. Let $Z(s(r), u^*)$ be the quantity of the composite good for which $U[Z(s(r), u^*), s(r)] = u^*$. In consequence, we want to minimize the function

$$C = \int_0^{\widehat{r}} [T(r) + Z(s(r), u^*) + R_A s(r)]\, n(r) dr \tag{29}$$

subject to the land constraint

$$s(r)n(r) = 2\pi r \qquad \text{for all } r \leq \widehat{r} \tag{30}$$

and the population constraint

$$\int_0^{\widehat{r}} n(r)dr = N. \tag{31}$$

Using (30) and (31), we readily verify that minimizing (29) amounts to solving the following maximization problem:

$$\max_{\widehat{r},s(r)} S = 2\pi \int_0^{\widehat{r}} \left[\frac{Y - T(r) - Z(s(r), u^*)}{s(r)} - R_A\right] rdr$$

subject to (31) in which $n(r) = 2\pi r/s(r)$.

Neglecting for the moment the population constraint, we may solve this problem by maximizing $[Y - T(r) - Z(s(r), u^*)]/s(r)$ with respect to $s(r)$ at each $r \leq \widehat{r}$. By definition of $S(r, u^*)$, it must be that the efficient land consumption $s(r)$ is identical to the equilibrium land consumption of land for each $r \leq \widehat{r}$, a condition which holds if and only if

$$\frac{Y - T(r) - Z[s(r), u^*]}{s(r)} = \Psi(r, u^*) \qquad \text{for all } r \leq \widehat{r}$$

where $\Psi(r, u^*)$ is the bid rent given by (15). Therefore, in order to maximize S, $\widehat{r}$ must satisfy

$$\Psi(r, u^*) = R_A$$

because $\Psi(r, u^*)$ is decreasing in r. Since this equation has a unique solution, it must be that $\widehat{r} = r^*$. Given (28), it is easily seen that $(s(r), \widehat{r})$ satisfies the population constraint (31) because $s(r) = S(r, u^*)$ and $\widehat{r} = r^*$. Consequently, we may conclude as follows:

Proposition 5 *The residential equilibrium is efficient.*

3.4 Social Stratification and Amenities

In the Thünian model, we have seen that the presence of intermediate goods gives rise to two types of configurations, segregated or integrated (Section 3.2.2). A related question in understanding the working of a city is to determine how consumers with different incomes organize themselves within the city. As in the Thünian model, *each location is occupied by the consumers with the highest bid rent.* Therefore, land being a normal good, we will not observe an integrated configuration because consumers endowed with different incomes have different bid rent functions and there is no direct interaction among them (e.g., home services from the poor to the rich). Hence, the residential equilibrium involves segregation. What remains to be determined, however, is the shape of the corresponding *social stratification* at the residential equilibrium. We will see in the last section that an integrated configuration involving households and farmers may emerge when the former value the rural amenities generated by the latter.

3.4.1 Why is downtown Detroit poor?

Consider the simple case of a finite number m of income classes with N_i consumers in class i; without loss of generality, we assume that $Y_1 < Y_2 < .... < Y_m$. All consumers have the same preferences $U(s, z)$, face the same commuting costs $T(r)$, and are a priori indifferent about their residential location. Replacing Y with Y_i in (15), we denote by $\Psi_i(r, u_i^*)$ the bid rent function of the i-th income class and by $S_i(r, u_i^*)$ the associated land consumption. It then follows from (18) that

$$\frac{d\Psi_i(r, u_i^*)}{dr} = -\frac{T^{'}(r)}{S_i(r, u_i^*)} < 0$$

for all $i = 1, ..., m$. Because a given group will occupy the area of the city where it outbids the other groups, the social stratification results from the ranking of the bid rent functions in terms of their slope in a sense that will now be defined.

If the social groups $j < k$ occupy adjacent plots, the land rent must the same for the two groups at the boundary $\overline{r}$ separating them. As a consequence, wherever the two bid rent curves $\Psi_j(r, u_j^*)$ and $\Psi_k(r, u_k^*)$ intersect at $\overline{r} \geq 0$, (17) and the normality of land imply the following inequality:

$$S_j(\overline{r}, u_j^*) \equiv \widehat{s}\left[\Psi_j(\overline{r}, u_j^*), Y_j - T(\overline{r})\right] < \widehat{s}\left[\Psi_k(\overline{r}, u_k^*), Y_k - T(\overline{r})\right] \equiv S_k(\overline{r}, u_k^*).$$

Hence, by (18), $\Psi_j(r, u_j^*)$ is steeper than $\Psi_k(r, u_k^*)$ at $\overline{r}$:[15]

$$\Delta = -\frac{T^{'}(\overline{r})}{S_j(\overline{r}, u_j^*)} + \frac{T^{'}(\overline{r})}{S_k(\overline{r}, u_k^*)} < 0.$$

[15] Note that the lot size is discontinuous at the border between two adjacent social areas. This corresponds to the discontinuity observed in the employment level at the border between two adjacent zones of production in the neoclassical model of land use of Section 3.2.2.

In other words, consumers of class j (k) will outbid those of class k (j) on the left (right) side of $\overline{r}$. Repeating the same argument for each pair (j, k) of income classes, we find that the N_1 consumers of the lowest income class occupy a disk of land centered at the CBD, the N_2 consumers with the second-lowest income occupy a ring surrounding this disk,..., and the N_m consumers belonging to the richest class are situated in the outermost ring. Thus, we have the following:

Proposition 6 *Assume that consumers have the same preferences and commuting cost function. Then, the social stratification of consumers within the city obeys the rule of concentric rings such that the consumer classes are ranked by increasing income as the distance from the CBD rises.*

Although some American central cities have rich enclaves, this result sheds light on the stylized fact that, in many U.S. cities, the poor live near the city center and the wealthy in the suburbs. Proposition 3.6 also offers a new perspective into the political economy of the city. An increase in the income of the rich consumers relaxes competition for land because these consumers move farther away from the center, making all income groups better off. On the other hand, raising the income of the poor consumers intensifies competition for land and pushes the rich farther away in the suburbs; eventually the poor people are better off but the rich ones are worse off. This suggests a potential conflict between the two classes: the poor have no objection to the rich class becoming richer, but the latter may find it better to keep the poor class poor. This agrees with the fact that shocks in the income distribution induce the development of particular urban sections at the expense of others, whereas the rich class members often try to lobby urban governments to implement restrictive zoning policies.

Having said this, we must acknowledge that this proposition is far from providing a complete answer to the stratification problem. Neglected factors governing the distribution of consumers over the urban space include the size of the family, the value of commuting time, and the financial support of the school system. Although we will not study these factors exhaustively, their impact can be summarized as follows:

1. A larger family has a stronger preference for space, which makes it live farther away from the CBD to benefit from the lower land rent prevailing there (Beckmann 1973).

2. The above proposition relies on the assumption that all consumers face the same commuting costs so that their share in consumers' expenditure

falls with their income. If higher income workers place a higher value on their commuting time, they face a trade-off between a higher land demand (due to normality of land) and the extra value of commuting time. As a result, the low-income consumers reside near the center and the middle class consumers in the suburbs; however, now the high-salary professionals and working couples choose to reside close to the CBD, because of their high value of time, in an urban section different from that of the poor consumers (Fujita 1989, chap. 2).

To illustrate, consider two income groups, the rich and the poor ($m = 2$ and $Y_1 < Y_2$). Rich households have a higher opportunity cost of time, and thus a higher commuting cost per mile, than the poor. Therefore, the rich value the accessibility to the CBD more than the poor. The net effect of these forces hinges on the behavior of the ratio of marginal commuting cost per mile and housing consumption. For example, if workers' commuting costs are proportional to their incomes, $T(r)Y_i$, we have

$$\Delta = -\frac{T'(\overline{r})Y_1}{S_1(\overline{r}, u_1^*)} + \frac{T'(\overline{r})Y_2}{S_2(\overline{r}, , u_2^*)}$$

which is positive if the income share spent on housing at $\overline{r}$ by the poor exceeds that of the rich:

$$\frac{R(\overline{r})S_1(\overline{r}, u_1^*)}{Y_1} > \frac{R(\overline{r})S_2(\overline{r}, u_2^*)}{Y_2}.$$

In this case, the rich locate near the CBD and the poor in the outer ring.

3. Decentralizing the supply of local public goods within a city strengthens the spatial sorting of households based on income differences. Indeed, high-income consumers can afford to pay for high-quality public services provided in districts populated by households having similar socioeconomic characteristics, while low-income consumers can pay only for low-quality services (Henderson and Thisse, 2001). In particular, when the financing of education is decentralized, families valuing more education (who are often those with higher incomes) similarly cluster in order to supply a better education to their offsprings. This results in higher human capital in the corresponding neighborhoods, thus perpetuating social and spatial segregation because the integrated equilibrium is unstable whereas the segregated one is stable (Bénabou 1994).

3.4.2 Why is central Paris rich?

Many European cities, such as London, Paris, Barcelona or Rome, display a social stratification that vastly differs from the one observed in the U.S.: the high-income people are located by the city center and the poor in the outer suburbs (Hohenberg and Lees, 1995).[16] This difference may be explained by the fact that those cities have well-preserved historical centers (Bruekner, Thisse and Zenou 1999).

Historical amenities are generated by monuments, buildings, parks, and other urban infrastructure from past eras that are pleasing to residents. Because these amenities are exogenous, they can be viewed as a causal factor in determining the pattern of location. In this case, households display a "love for city center," and thus their preferences depend on housing consumption s and consumption of a composite good z, as well as on amenities $a(r)$, which is viewed as a local public good subject to a distance-decay effect.

Differentiating the spatial equilibrium condition and using the envelope theorem yields the expression:

$$-T^{'}(r) - \frac{d\Psi_i(r, u_i^*)}{dr} S_i(r) + \frac{\partial V}{\partial a}\frac{da(r)}{dr} = 0$$

and thus

$$\frac{d\Psi_i(r, u_i^*)}{dr} = \frac{1}{S_i(r, u_i^*)}\left[-T^{'}(r) + \frac{\partial V}{\partial a}\frac{da(r)}{dr}\right] \tag{32}$$

where the marginal value of amenities $\partial V/\partial a$ is evaluated *after* the optimal adjustment of housing consumption. Hence, when amenities decline with distance, the land rent must compensate for inferior amenities as well as the high cost of commuting.

Using (32), we readily verify that the difference Δ between the slopes of the bid rent functions at $\overline{r}$ for $j < k$ is given by

$$\Delta = -\frac{T^{'}(\overline{r})}{S_j(\overline{r}, u_j^*)} + \frac{T^{'}(\overline{r})}{S_k(\overline{r}, u_k^*)} + \frac{1}{S_j(\overline{r}, u_j^*)}\left.\frac{\partial V}{\partial a}\right|_j \frac{da(\overline{r})}{dr} - \frac{1}{S_k(\overline{r}, u_k^*)}\left.\frac{\partial V}{\partial a}\right|_k \frac{da(\overline{r})}{dr}. \tag{33}$$

For the social stratification obtained in Proposition 3.6 to be revered, this expression must be positive. This will be so if historical landmarks appeal to high-income and educated households. For example, Glaeser (2011) observes

[16] Ingram and Carroll (1981) show that this pattern also exists in a number of Latin American cities.

that New Yorkers who live in the historic districts of Manhattan are on average 70 per cent wealthier than those who live outside such areas. With such numbers in mind, we find it reasonable to assume that the marginal valuation of historic amenities (after adjustment of s) rises with income faster than housing:[17]

$$\frac{1}{S_j(\overline{r}, u_j^*)} \left.\frac{\partial V}{\partial a}\right|_j - \frac{1}{S_k(\overline{r}, u_k^*)} \left.\frac{\partial V}{\partial a}\right|_k < 0.$$

If $da(r)/dr < 0$ is small in absolute value, then the entire amenity term in (33) is positive but close to zero. The negative sign of the first two terms of (33) (recall that $S_i(\overline{r}) < S_j(\overline{r})$) will then dominate. Thus, if the center's amenity advantage over the suburbs is weak, the U.S. location pattern holds: the poor live in the center and the rich live in the suburbs.

On the other hand, if $da(r)/dr$ is large in absolute value, then the amenity term dominates the conventional forces in determining the sign of Δ. In other words, if the center has a large amenity advantage, so that amenities fall rapidly with distance, then the U.S. pattern is reversed: the rich live in the city center and the poor live in the suburbs. This corresponds to the case of Paris, which has a steep amenity gradient and central location of the rich. In contrast, since an American urban area like Detroit lacks the rich history of Paris, the central-city's infrastructure does not offer appreciable aesthetic benefits. This means that no amenity force is working to reverse the conventional forces that draw the rich to the suburbs. As a result, central Detroit is poor.

Superior amenities make the central city rich, while weak amenities make it poor. Because location by income is now linked to a city's idiosyncratic features, the multiplicity of observed location patterns around the world becomes explicable. In particular, Europe's longer history provides an obvious reason why its central cities contain more buildings and monuments of historical significance than do their U.S. counterparts. Many European cities were major metropolises at a time when much of the U.S. had not even been settled, and the legacy of urban development from this distant past provides an atmosphere in European city centers that appears to be highly valued by the residents. In addition to the effect of a longer history, government

[17] This assumption is consistent with familiar specifications of preferences. For example, Bruekner et al. (1999) shows that it holds under CES preferences when the elasticity of substitution between goods exceeds 1.

investment in central-city infrastructure appears in many cases to have been more extensive in European cities than in the U.S.

Note also that the pattern of exogenous amenities differs across cities because a central city's historical amenities are determined mainly by past government decisions regarding investments in urban infrastructure. However, although the concept itself might suggest otherwise, historical amenities depreciate over time, which means that their maintenance requires ongoing investment. If such expenditures were withheld, the central city's amenities would decay, and high-income residents would be increasingly drawn to the suburbs.

3.4.3 Why to live in the sticks?

We turn to a different issue in which amenities also plays a major role. In several countries, the last decades have witnessed the emergence of a new pattern of urban development called the "periurban belt," that is, a zone surrounding the city and occupied both by commuting workers and farmers. As a result, a periurban belt may be viewed as a rural space in the sense that the majority of its land is used for farming purposes, as well as an urban space with most of its working population commuting to the city. In 1999, periurban areas cover 33% of France and accommodate 21% of its population.

The main reason for the existence of such an integrated space is that *consumers value the rural amenities created by farmers as well as the greenness of the environment* (Cavailhès et al., 2004). For a periurban belt to arise, consumers' and farmers' bid rents must be the same over the belt. Such an equality is possible because the amenity level at a given location is determined through the interactions between consumers and farmers whose respective densities are endogenous.

Consumers share Cobb-Douglas preferences:

$$U(z, s, a) = z^{\alpha} s^{\beta} a^{\gamma} \qquad \alpha, \beta, \gamma > 0 \quad \text{and} \quad \alpha + \beta = 1 \tag{34}$$

where a is the amount of amenities available at the consumer's location. For simplicity, the level of urban amenities available within the city is assumed to be uniform and constant (a is normalized to 1). By contrast, the level of rural amenities available in the periurban belt is endogenous.

Farmers produce under constant returns and sell their product to the food processing industry, thus implying that the cost of land R_A is positive.

Because the distance to the food processing plants is immaterial for our purpose, we may assume that R_A is constant. Rural amenities $a(r)$ are a by-product of farming, the level of which is a linear function of the total area used for farming:

$$a(r) = \delta n_f(r) s_f(r) \qquad \delta > 0 \tag{35}$$

where n_f is the number of farmers and s_f the individual surface they use.

The total amount of space occupied by consumers and farmers at r is equal to the supply of land (which is normalized to 1):

$$n_c(r)s_c(r) + n_f(r)s_f(r) = 1 \tag{36}$$

where n_c is the number of residents and s_c the size of a residential plot.

For r to belong to a periurban area, it must be that $a(r) > 1$; otherwise consumers would prefer living in the city. Given (36) and (35), this condition requires $\delta > 1$. In other words, farming must produce a sufficiently large amount of rural amenities for a periurban area to exist.

To simplify the expressions of consumers' bid rent functions, we assume that the utility level is given by the outside option households face ($\bar{u} = 1$) while allowing the population size to be variable (the open city model discussed in the concluding section). It is then readily verified that the bid rent function of a consumer residing in a periurban belt is as follows:

$$\Psi_p(r) = (Y - tr)^{1/\beta} \left[a(r)\right]^{\gamma/\beta}$$

whereas the bid rent of a city consumer is given by

$$\Psi_c(r) = (Y - tr)^{1/\beta} . \tag{37}$$

For farmers and consumers to share land at r, their bid rents must be the same:[18]

$$\Psi_p(r) = R_A$$

which implies that the equilibrium amount of rural amenities available at r is such that

$$a^*(r) = \left[\frac{R_A}{\Psi_c(r)}\right]^{\beta/\gamma} .$$

[18] See Section 3.2.2 for a similar case.

Rural amenities thus shift upward consumers' bid rent to the level of farmers' bid rent. The above expression also implies that $a^*(r)$ increases with r, which means that the size of the farming area must increase with the distance to the city center. This is because consumers need a higher level of rural amenities to be compensated for their longer commuting. This in turn implies that the size of the residential area decreases with the distance to the CBD. The periurban belt vanishes when all the land is used for farming.

When the income Y is sufficiently large, (37) implies that $\Psi_c(0) > R_A$. Because $\Psi_c(r)$ is strictly decreasing with $\Psi_c(Y/t) = 0$, $a^*(r)$ is increasing and the equation $a^*(r) = 1$ has a unique solution $r_u > 0$. Therefore, $[0, r_u]$ is occupied by consumers only, whereas the periurban belt starts from r_u and ends at r_p where land is used for agricultural activities only, i.e. $a^*(r_p) = \delta$ (see Figure 3.5 for an illustration).

Figure 3.5: The equilibrium land rent under rural amenities

A few remarks are in order. First, at the same distance r from the city center, the residential plot is smaller in the periurban belt than in the city that does not possess such a belt. Indeed, because the land rent in the periurban belt capitalizes the rural amenities, it is higher than in a purely urban area, thus reducing the consumption of land. Furthermore, whereas the lot size increases within the city as one moves away from the center, it decreases inside the periurban belt, while consumers bear higher commuting costs and pay a land rent that remains constant and equal to R_A. These results do not contradict what we have seen above because consumers are compensated by more rural amenity as the distance to the CBD rises. Last, rural amenities having the nature of a local public good available at r, their Lindhal price $P(r)$ is given by the marginal rate of substitution between rural amenities and the composite good z, times the number of consumers residing there:

$$P(r) = \frac{\partial U/\partial a}{\partial U/\partial z} n_c^*(r) = \frac{\gamma}{\beta}\Psi_p(x)\left[\frac{1}{A_p^*(x)} - \frac{1}{\delta}\right].$$

As expected, the equilibrium price of the rural amenities depends positively on the land rent in the periurban belt and negatively on the amount of available amenities. Everything else being equal, the higher γ/β, the higher the amenity price. Observe also that $P(r) = 0$ when $\gamma = 0$, i.e. when consumers do not value rural amenities.

3.5 Discrete Foundations of Continuous Land Use Theory

The monocentric city model differs from standard microeconomics in that all the unknowns are described by *density* functions. Instead, one might want to develop a discrete model with a finite number of households each consuming a positive amount of the composite good as well as a positive amount of land. Though Alonso himself has proposed two alternative formulations of such a discrete model, very little work has been devoted to this issue.[19] In this section, we follow Asami, Fujita, and Smith (1990) as well as Berliant and Fujita (1992) and study a simple one-dimensional model in which consumers are identical.

Space is described by the interval $X = [0, \infty)$ with a unit density of land everywhere and the CBD located at the origin; the opportunity cost of land R_A is constant and positive. A finite number n of consumers may be accommodated in this area. The utility of a consumer is $U(z, s)$ where z is the quantity of the composite good, but now $s > 0$ is the size of a lot defined by an interval $[r, r + s) \subset X$. If a consumer occupies the lot $[r, r + s)$, r describes his location, and the commuting cost is defined by tr where t is a positive constant. All consumers have the same income Y and the same utility function U, which satisfies all the properties stated in Section 3.3.1. As in Section 3.3.1, let $Z(s, u)$ be the positive quantity of the composite good that yields utility level u when a consumer occupies a lot of size $s > 0$. Recall that $Z(s, u)$ is strictly decreasing, strictly convex and such that $\lim_{s \to 0} Z(s, u) = \infty$.

An allocation $(z_i, s_i, r_i;\ i = 1, ..., n)$ is defined by a consumption bundle and a location for each consumer. It is feasible if and only if no pair of lots overlap. Without loss of generality, we may rank consumers such that $r_1 < r_2 < ... < r_n$.

Let $R(r)$ be the land price function defined on X such that a consumer choosing a lot $[r, r + s)$ pays $R(r)s$ for the lot. Then the consumer problem

[19] By contrast, the connection between the Arrow-Debreu model and the continuous approach to general equilibrium developed by Aumann has attracted a lot of attention in microeconomics (Hildenbrand 1974).

is given by

$$\max_{r,z,s} U(z,s) \qquad \text{s.t.} \quad z + R(r)s = Y - tr$$

This is formally identical to (13) where $T(r) = tr$. Therefore, if this consumer chooses location r and achieves the utility level u, then he must choose a lot size $S(r,u)$ which maximizes the bid rent function (15).

A residential equilibrium with n consumers is given by a utility level u^*, a land price function $R^*(r)$ together with a feasible allocation $(z_i^*, s_i^*, r_i^*; i = 1, ..., n)$ such that the following conditions hold:

$$R^*(r) \geq \max\{\Psi(r, u^*), R_A\} \tag{38}$$

$$R^*(r_i^*) = \Psi(r_i^*, u^*) \qquad i = 1, ..., n \tag{39}$$

$$R^*(r_n^*) = R_A \tag{40}$$

$$s_i^* = S(r_i^*, u^*) \qquad i = 1, ..., n \tag{41}$$

$$r_1^* = 0 \quad \text{and} \quad r_{i+1}^* = r_i^* + s_i^* \qquad i = 1, ..., n-1 \tag{42}$$

where the bid rent function $\Psi(r, u^*)$ is defined by (15) for all $r \geq 0$. The condition (40) on the land rent for the last consumer allows us to avoid unnecessary technical difficulties;[20] condition (42) states that there is no vacant land within the city.

Clearly, since the bid rent function decreases with distance, the equilibrium rents R_i^* satisfy $R_1^* > R_2^* > ... > R_n^* = R_A$. Furthermore, since $S(r, u^*)$ is strictly decreasing in r (Proposition 3.4.(ii)) $i < j$ implies that $s_i^* < s_j^*$. So, we have shown:[21]

Proposition 7 *Consider any finite number of consumers and assume that a residential equilibrium exists. Then, this residential equilibrium is such that the land rent decreases as one moves away from the CBD whereas consumers with larger lots locate farther from the CBD than consumers with smaller lots.*

[20] When this condition is replaced by the inequalities $\Psi(r_n, u^*) \geq R_A$ and $\Psi(r_n + s_n, u^*) \leq R_A$, there is a continuum of equilibria (Asami et al. 1990, Theorem 2).

[21] Standard tools of general equilibrium analysis are not applicable here due to nonconvexities. However, existence, uniqueness and optimality have been shown by Asami et al. (1990).

This means that the residential equilibrium with a finite number of consumers displays the same basic features as the continuous standard model of urban economics. However, the preceding discrete model suffers from a serious defect, namely, each consumer pays the same price for each unit of his lot, and thus the landowner may want to extract more from the consumer or a consumer may buy more land for resale to the next one. To avoid this difficulty, one may either assume that arbitrage is prohibitively costly or that a consumer located at r pays a price given by

$$\int_r^{r+s} R(y)dy$$

for the lot $[r, r+s)$, which is also suggested by Alonso. Proposition 3.7 remains essentially the same in this alternative model, but the analysis is more complex (Berliant and Fujita 1992).

Note, finally, that Asami, Fujita, and Smith (1990) have shown that the standard continuous model provides a good approximation of the discrete model considered in this section when n is large enough. In particular, a finite economy with n consumers and a continuous economy with a mass N of consumers each have a cumulative population distribution; these authors show (Theorem 5) that the two sequences of normalized (by the population size) distributions have the same limit as $n = N$.

3.6 Notes on the Literature

Two fundamental ideas lie at the heart of urban economics: (i) people prefer shorter trips to longer trips and (ii) people prefer having more space than less space. The urban land use model presented in Section 3.3.1 encapsulates this trade-off within a simple framework, which has been developed independently by Beckmann (1957; 1969), Mohring (1961), Alonso (1960; 1964), Muth (1961; 1969), Mills (1967; 1972b), Casetti (1971), and Solow (1973). However, it seems fair to say that Beckmann's short article published in the first issue of the *Journal of Economic Theory* and based on his *1957* discussion paper is really path-breaking in that it is not only a concise statement of the standard monocentric model but also a precursor to several later contributions.

Definitions of closed and open cities were introduced by Wheaton (1974), whereas the public ownership model is credited to Solow (1973). The existence of a residential equilibrium with a continuum of locations and a single marketplace for heterogeneous consumers has been established by Fujita and Smith (1987). The study of the monocentric model when consumers have heterogeneous tastes as described by the logit can be found in Anas (1990), whereas the extension of the standard model to several prespecified centers was considered by Papageorgiou and Casetti (1971).

The comparative statics of the residential equilibrium was first studied by Wheaton (1974) in the case of homogeneous consumers. The optimal city was studied by Mirrlees (1972), who used a utilitarian welfare function. The approach taken in Section 3.3.3 is based on Herbert and Stevens (1970), who retained a discrete space and used the duality theorem of linear programming. The general analysis of the residential equilibrium with several income classes has been presented by Hartwick, Schweizer, and Varaiya (1976). Finally, the discrete foundations of the continuous urban model have been criticized by Berliant (1985). Possible solutions have been investigated by Asami et al. (1990), Papageorgiou and Pines (1990), and Berliant and Fujita (1992).

The state of the art in urban economics is summarized in the three complementary books by Fujita (1989), Papageorgiou and Pines (1999) and Zenou (2009), whereas a historical and methodological outlook of this field is provided by Baumont and Huriot (2000).

4 CONCLUDING REMARKS

We have seen how land use patterns and land rent profiles can be determined in competitive land markets once it is assumed that a center exists where (some or all) tradable goods have to be shipped. As in Chapter 2, we have assumed that there are no physical differences in land at different locations. The differences in land rents can therefore be attributed to the relative advantage of each location compared with the extensive margin of land use. In other words, the land rent at a given location corresponds to a *locational rent*.[22] Since the city is the reason for this locational rent to come into being, it should be clear that the critical issue is now to figure out why cities, or

[22] The concept of locational rent is to be contrasted to the more standard concept of scarcity rent, which could be integrated into the Thünian model by assuming that the "isolated state" is replaced by a "small circular island." At the border of the island, the land rent would be positive, and this value would express the global scarcity of land, whereas the difference in the value of the land rent inside the island would still have the nature of a locational rent.

central business districts, exist. This question has haunted geographical economics for decades. We will see in different chapters of this book how various economic and non-economic reasons may explain why people choose to form a city.

In the Thünian model, the land rent is equal to the excess of revenues obtained from the sale of goods produced by using land over payments to nonland factors used in production and transportation. This is why the bid rent function is obtained from a condition of zero profits, which can be interpreted as a free-entry condition of producers in each activity considered. When consumers (instead of producers) are the land-users, the mechanism leading to the formation of the bid rents is similar to the one uncovered by Thünen provided that the utility level is given by the reservation utility and the population size is variable (this amounts to a condition of free entry). This is called the *open city* model, an example of which will be discussed in the next chapter. By contrast, in the *closed city* model in which the population size is fixed, the utility level is endogenous. This requires a more general approach to the formation of bid rents such as that studied in Section 3.3.1. Though both types of models (closed city and open city) lead to similar results, they are useful because they correspond to different situations.

In addition, we have implicitly assumed absentee landlords. That is, the land rent earned within the city goes to landlords who do not reside within the city; hence, the rent does not feed back into consumers' incomes. Both the closed city and open city models can be extended to cope with public ownership of land in which the aggregate land rent is first collected by a public agency and then equally shared among consumers. The analysis remains essentially the same. The choice of a particular specification (open versus closed, absentee landlords versus public property) is dictated by the main features of the problem under consideration. In this chapter, we have chosen to present the most popular model, and we refer to Fujita (1989) for more details regarding the other approaches. Last, the market-town in Section 3.2 and the CBD in Section 3.3 are exogenously given. We will see in Chapters 6 and 10 how they can be determined endogenously.

A word, before closing. We have seen that housing and commuting costs increase with the city size. Every else being equal, this reduces the level of consumers' real income, and thus makes the city less attractive. In other words, *land use appears to be a major dispersion force in the making of the space-economy.* The intensity of this force varies inversely with the efficiency of the technology used by commuters and the supply of transportation infrastructures. We will see later in this book how land use interacts with various agglomeration forces to shape the economic landscape.